ROTATION OF TENSORS IN THE X-Y PLANE

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References: Kip S. Thorne & Roger D. Blandford, *Modern Classical Physics*, Princeton University Press (2017). Exercise 1.6.
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Consider a 2-d Cartesian system as shown in Fig. 1. The original basis vectors are shown in black and the rotated axes in orange. The coordinates of the rotated x axis (denoted $\overline{x}$) are $(\cos\phi, \sin\phi)$ and those of the rotated y axis $\overline{y}$ are $(-\sin\phi, \cos\phi)$ as shown. That is, we have for the basis vectors

$$\begin{aligned} \boldsymbol{e}_{\overline{x}} &= \cos\phi\ \boldsymbol{e}_x + \sin\phi\ \boldsymbol{e}_y \\ \boldsymbol{e}_{\overline{y}} &= -\sin\phi\ \boldsymbol{e}_x + \cos\phi\ \boldsymbol{e}_y \end{aligned} \tag{1}$$

Ex 1.6(a) To convert the barred axes back to the original axes, we want a matrix R such that

$$\begin{aligned} \boldsymbol{e}_x &= R\boldsymbol{e}_{\overline{x}} \\ \boldsymbol{e}_y &= R\boldsymbol{e}_{\overline{y}} \end{aligned} \tag{2}$$

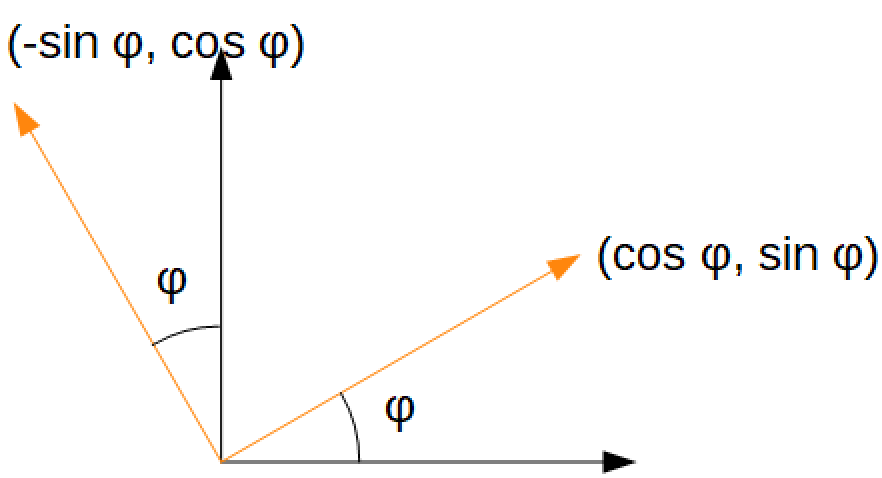

FIGURE 1. A 2-d Cartesian system and its rotation.

or, writing out in components

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\phi \\ \sin\phi \\ 0 \end{bmatrix} \quad (3)$$

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{bmatrix} \quad (4)$$

By inspection, we see that the required rotation matrix is

$$R = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (5)$$

The inverse is just the transpose, as we can verify by direct multiplication

$$R^T R = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (6)$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (7)$$

Note that the inverse R^{-1} is obtained by replacing ϕ by $-\phi$, which makes sense since to invert the rotation we rotate by the same amount in the opposite direction.

Ex 1.6(b) Now consider the representation of a given point $\mathscr{P}$ in both the original and rotated systems. In the original system we have

$$\mathscr{P} = x\boldsymbol{e}_x + y\boldsymbol{e}_y \quad (8)$$

and in the rotated system we have

$$\mathscr{P} = \bar{x}\boldsymbol{e}_{\bar{x}} + \bar{y}\boldsymbol{e}_{\bar{y}} \quad (9)$$

We can use the same symbol $\mathscr{P}$ to refer to the point in both coordinate systems, since it's a geometric object, independent of coordinate system.

We can substitute 1 into 9 to get

$$\mathscr{P} = \bar{x}\left(\cos\phi\ \boldsymbol{e}_x + \sin\phi\ \boldsymbol{e}_y\right) + \bar{y}\left(-\sin\phi\ \boldsymbol{e}_x + \cos\phi\ \boldsymbol{e}_y\right) \quad (10)$$

$$= \left(\bar{x}\cos\phi - \bar{y}\sin\phi\right)\boldsymbol{e}_x + \left(\bar{x}\sin\phi + \bar{y}\cos\phi\right)\boldsymbol{e}_y \quad (11)$$

Therefore, we have

$$\begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \overline{x} \\ \overline{y} \\ 0 \end{bmatrix} \tag{12}$$

$$= R^{-1} \begin{bmatrix} \overline{x} \\ \overline{y} \\ 0 \end{bmatrix} \tag{13}$$

Multiplying both sides by R gives the inverse rotation:

$$\begin{bmatrix} \overline{x} \\ \overline{y} \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \tag{14}$$

It's also possible to obtain the transformation by considering the geometry of the rotation, but it's considerably more involved.

Ex 1.6 (c) We now consider a vector A_j. T&B use the electromagnetic vector potential as the vector, but the argument works for any vector. We rotate the quantity $A_x + iA_y$ using 14. I'll use the shorthand notation

$$\begin{aligned} c &\equiv \cos\phi \\ s &\equiv \sin\phi \end{aligned} \tag{15}$$

We get

$$A_{\overline{x}} + iA_{\overline{y}} = cA_x + sA_y - isA_x + icA_y \tag{16}$$

$$= A_x(c - is) + A_y(s + ic) \tag{17}$$

$$= (c - is)(A_x + iA_y) \tag{18}$$

$$= (A_x + iA_y)e^{-i\phi} \tag{19}$$

Ex 1.6 (d) Now consider a rank-2 symmetric tensor with zero trace, and confined to the $x-y$ plane. These conditions require all components with a z index to be zero, and further:

$$\begin{aligned} h_{xx} &= -h_{yy} \\ h_{xy} &= h_{yx} \end{aligned} \tag{20}$$

To rotate a rank-2 tensor, we can use a scaled down version of T&B's equation 1.13b, so we have

$$h_{\overline{pq}} = R_{\overline{p}i} R_{\overline{q}j} h_{ij} \tag{21}$$

where $R_{\overline{p}i}$ and $R_{\overline{q}j}$ are elements of the matrix R in 5. Writing out the double sum, we have

$$\begin{aligned} h_{\overline{xx}} + ih_{\overline{xy}} = c\left(ch_{xx} + sh_{xy}\right) + s\left(ch_{yx} + sh_{yy}\right) + \\ ic\left(-sh_{xx} + ch_{xy}\right) + is\left(-sh_{yx} + ch_{yy}\right) \end{aligned} \tag{22}$$

We now apply the conditions 20 to get

$$h_{\overline{xx}} + ih_{\overline{xy}} = h_{xx}\left(c^2 - isc - s^2 - isc\right) + h_{xy}\left(sc + sc + ic^2 - is^2\right) \tag{23}$$

$$= h_{xx}\left(c^2 - s^2 - 2isc\right) + ih_{xy}\left(c^2 - s^2 - 2isc\right) \tag{24}$$

We now use the trig identities

$$c^2 - s^2 = \cos^2\phi - \sin^2\phi = \cos 2\phi \tag{25}$$

$$2sc = 2\sin\phi\cos\phi = \sin 2\phi \tag{26}$$

and we get

$$h_{\overline{xx}} + ih_{\overline{xy}} = \left(h_{xx} + ih_{xy}\right)\left(\cos 2\phi - i\sin 2\phi\right) \tag{27}$$

$$= \left(h_{xx} + ih_{xy}\right)e^{-2i\phi} \tag{28}$$