PROPERTIES OF THE LEVI-CIVITA TENSOR

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References: Kip S. Thorne & Roger D. Blandford, *Modern Classical Physics*, Princeton University Press (2017). Exercise 1.7.

Post date: 20 Sep 2020.

T&B define the totally antisymmetric Levi-Civita tensor ϵ as the volume of a parallelepiped, where the vectors defining the edges of the parallelepiped are inserted into the tensor's slots. That is

$$\epsilon(\boldsymbol{A},\boldsymbol{B},\ldots,\boldsymbol{F}) \equiv \text{volume of paralellepiped} \tag{1}$$

where $\boldsymbol{A},\boldsymbol{B},\ldots,\boldsymbol{F}$ are the edges. For an n-dimensional parallelepiped with orthonormal edges, the volume must have magnitude 1, although the order in which the edges are inserted into the slots can result in ±1 as the actual volume. For example, in 2 dimensions, the volume becomes the area of a square with side lengths of 1, and the area is +1 if the edges are inserted in the order $\boldsymbol{e}_x,\boldsymbol{e}_y$ but -1 if the edges are in the order $\boldsymbol{e}_y,\boldsymbol{e}_x$.

We can see that ϵ is zero if the collection of edges is linearly dependent, that is, if one or more of the edges can be written as a linear combination of the remaining edges. This follows from the linear dependence of a tensor on the vectors that are inserted into its slots. For example, consider the rank-3 tensor

$$\epsilon(\boldsymbol{A},\boldsymbol{B},\boldsymbol{A}+\boldsymbol{B}) = \epsilon(\boldsymbol{A},\boldsymbol{B},\boldsymbol{A}) + \epsilon(\boldsymbol{A},\boldsymbol{B},\boldsymbol{B}) \tag{2}$$

Since ϵ is antisymmetric under interchange of any two of its vectors, we see that

$$\epsilon(\boldsymbol{A},\boldsymbol{B},\boldsymbol{A}) = -\epsilon(\boldsymbol{A},\boldsymbol{B},\boldsymbol{A}) = 0 \tag{3}$$

$$\epsilon(\boldsymbol{A},\boldsymbol{B},\boldsymbol{B}) = -\epsilon(\boldsymbol{A},\boldsymbol{B},\boldsymbol{B}) = 0 \tag{4}$$

since any quantity equal to its negative is zero. This generalizes to any set of linearly dependent edges, since we can always decompose the tensor into a sum of terms, each of which will contain at least one pair of duplicated vectors.

Return now to the case where we have a linearly independent set of edges, so that $\epsilon \neq 0$. Once we have specified the value of ϵ for one combination of such edges, this determines the value of ϵ for any other combination of these same edges. For example, suppose we have a linearly independent set $\{\boldsymbol{A},\boldsymbol{B},\boldsymbol{C}\}$ and have defined the value of $\epsilon(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C})$. A linearly independent set formed by combining these 3 vectors is $\{\boldsymbol{A},\boldsymbol{B}+\boldsymbol{C},\boldsymbol{B}-\boldsymbol{C}\}$. We can find ϵ for this combination, using its linearity and antisymmetry properties:

$$\epsilon(\boldsymbol{A},\boldsymbol{B}+\boldsymbol{C},\boldsymbol{B}-\boldsymbol{C}) = \epsilon(\boldsymbol{A},\boldsymbol{B},\boldsymbol{B}-\boldsymbol{C})+\epsilon(\boldsymbol{A},\boldsymbol{C},\boldsymbol{B}-\boldsymbol{C}) \tag{5}$$

$$= \epsilon(\boldsymbol{A},\boldsymbol{B},\boldsymbol{B})-\epsilon(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C})+ \epsilon(\boldsymbol{A},\boldsymbol{C},\boldsymbol{B})-\epsilon(\boldsymbol{A},\boldsymbol{C},\boldsymbol{C}) \tag{6}$$

$$= 0-\epsilon(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C})+\epsilon(\boldsymbol{A},\boldsymbol{C},\boldsymbol{B})-0 \tag{7}$$

$$= -\epsilon(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C})-\epsilon(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C}) \tag{8}$$

$$= -2\epsilon(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C}) \tag{9}$$

In general, we can expand ϵ as a function of any linear combination of a linearly independent set into a sum of terms, some of which will contain duplicate vectors and are thus zero, and some of which will contain some permutation of the original linearly independent set. Terms of the latter type can be converted into the original order of the vectors by permuting the vectors in the slots, and each swap merely changes the sign, so we can always reduce an ϵ that is a function of any linearly independent set into some multiple of the original ϵ. Thus we require only one number to define the values of ϵ for all combinations of the vectors defining the parallelepiped.

From the definition 1 of ϵ as the volume of a parallelepiped and the other properties derived above, we see that the value of ϵ for a linearly independent set of orthonormal vectors must be ±1. By convention, in 3-d Euclidean space, the volume is defined to be +1 if the vectors $\boldsymbol{e}_x,\boldsymbol{e}_y,\boldsymbol{e}_z$ form a right-handed Cartesian coordinate system.

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