VECTOR IDENTITIES FOR THE CROSS PRODUCT AND CURL

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References: Kip S. Thorne & Roger D. Blandford, *Modern Classical Physics*, Princeton University Press (2017). Exercise 1.8.

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We can use the Levi-Civita tensor to write the 3-d vector cross product as (sum over j and k):

$$(\boldsymbol{A}\times\boldsymbol{B})_i = \epsilon_{ijk}A_jB_k \tag{1}$$

We can use this, together with the identity

$$\epsilon_{ijm}\epsilon_{k\ell m} = \delta^{ij}_{k\ell} \equiv \delta^i_k\delta^j_\ell - \delta^i_\ell\delta^j_k \tag{2}$$

to derive some common vector identities. 2 is a special case of the more general 4-d identity we derived earlier. To derive it in the 3-d Euclidean space case, we note that there is a sum over the index m. Each of ϵ_{ijm} and $\epsilon_{k\ell m}$ can therefore have only 2 non-zero terms, since all 3 indices must be different. For example, if $m=1$, then we can have $ij=23$ or $ij=32$ and similarly for $k\ell$. Thus either $i=k$ and $j=\ell$, in which case $\epsilon_{ijm}=\epsilon_{k\ell m}$ and $\epsilon_{ijm}\epsilon_{k\ell m}=(\pm1)^2=+1$, or $i=\ell$ and $j=k$, in which case $\epsilon_{ijm}=\epsilon_{\ell km}=-\epsilon_{k\ell m}$ and $\epsilon_{ijm}\epsilon_{k\ell m}=(+1)(-1)=-1$. These two cases give the RHS of 2.

Ex 1.8(a). We now work out some vector identities using 1 and 2.

Remember that a comma before an index indicates a derivative: $A_{k,j}=\frac{\partial A_k}{\partial x_j}$.

$$\begin{aligned}
[\nabla\times(\nabla\times\boldsymbol{A})]_\ell &= \epsilon_{\ell mi}\left(\epsilon_{ijk}A_{k,j}\right)_{,m} &(3)\\
&= \epsilon_{\ell mi}\epsilon_{ijk}A_{k,j,m} &(4)\\
&= \epsilon_{\ell mi}\epsilon_{jki}A_{k,j,m} &(5)\\
&= \delta^{\ell m}_{jk}A_{k,j,m} &(6)\\
&= \left(\delta^\ell_j\delta^m_k - \delta^\ell_k\delta^m_j\right)A_{k,j,m} &(7)\\
&= A_{k.\ell,k} - A_{\ell,m,m} &(8)\\
&= \left[\nabla(\nabla\cdot\boldsymbol{A}) - \nabla^2\boldsymbol{A}\right]_\ell &(9)
\end{aligned}$$

The last line follows because the order in which we take derivatives doesn't matter, so that

$$A_{k,\ell,k} = A_{k,k,\ell} \tag{10}$$

and

$$A_{k,k} = \nabla \cdot \boldsymbol{A} \tag{11}$$

Ex 1.8(b) We now have

$$\begin{aligned}
(\boldsymbol{A}\times\boldsymbol{B})\cdot(\boldsymbol{C}\times\boldsymbol{D}) &= \epsilon_{ijk}A_jB_k\epsilon_{i\ell m}C_\ell D_m &(12)\\
&= \left(\delta^j_\ell\delta^k_m - \delta^j_m\delta^k_\ell\right)A_jB_kC_\ell D_m &(13)\\
&= A_jB_kC_jD_k - A_jB_kC_kD_j &(14)\\
&= (\boldsymbol{A}\cdot\boldsymbol{C})(\boldsymbol{B}\cdot\boldsymbol{D}) - (\boldsymbol{A}\cdot\boldsymbol{D})(\boldsymbol{B}\cdot\boldsymbol{C}) &(15)
\end{aligned}$$

Ex 1.8(c) Finally, we have

$$\begin{aligned}
[(\boldsymbol{A}\times\boldsymbol{B})\times(\boldsymbol{C}\times\boldsymbol{D})]_a &= \epsilon_{ain}\left(\epsilon_{ijk}A_jB_k\epsilon_{n\ell m}C_\ell D_m\right) &(16)\\
&= -\epsilon_{ian}\epsilon_{ijk}A_jB_k\epsilon_{n\ell m}C_\ell D_m &(17)\\
&= \left(\delta^j_n\delta^k_a - \delta^j_a\delta^k_n\right)A_jB_kC_\ell D_m\epsilon_{nlm} &(18)\\
&= (A_nB_aC_\ell D_m - A_aB_nC_\ell D_m)\,\epsilon_{n\ell m} &(19)\\
&= B_aA_n(\epsilon_{n\ell m}C_\ell D_m) - A_aB_n(\epsilon_{n\ell m}C_\ell D_m) &(20)\\
&= [\boldsymbol{B}(\boldsymbol{A}\cdot(\boldsymbol{C}\times\boldsymbol{D})) - \boldsymbol{A}(\boldsymbol{B}\cdot(\boldsymbol{C}\times\boldsymbol{D}))]_a &(21)
\end{aligned}$$