LEVI-CIVITA TENSOR IN 2-DIMENSIONAL EUCLIDEAN SPACE

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

References: Kip S. Thorne & Roger D. Blandford, *Modern Classical Physics*, Princeton University Press (2017). Exercise 1.9.

Post date: 22 Sep 2020.

The Levi-Civita tensor is defined by T&B as the volume of a parallelepiped when it is applied to the parallelepiped's edges:

$$\epsilon(\boldsymbol{A},\boldsymbol{B},\ldots,\boldsymbol{F}) \equiv \text{volume of paralellepiped} \tag{1}$$

Ex 1.9(a) In 2-dim Euclidean space, we can define an orthonormal basis by the two vectors $\{\boldsymbol{e}_1,\boldsymbol{e}_2\}$ with components

$$\begin{aligned}\boldsymbol{e}_1 &= \begin{bmatrix}1\\0\end{bmatrix}\\ \boldsymbol{e}_2 &= \begin{bmatrix}0\\1\end{bmatrix}\end{aligned} \tag{2}$$

The volume (area) of the parallelogram (actually a square with side length 1) is then given by

$$\epsilon(\boldsymbol{e}_1,\boldsymbol{e}_2) = \epsilon_{ij}(\boldsymbol{e}_1)_i(\boldsymbol{e}_2)_j \tag{3}$$

The tensor ϵ is defined to be antisymmetric, so $\epsilon_{12} = -\epsilon_{21}$ and $\epsilon_{11} = \epsilon_{22} = 0$. We therefore have

$$\epsilon_{ij}(\boldsymbol{e}_1)_i(\boldsymbol{e}_2)_j = \epsilon_{12}(\boldsymbol{e}_1)_1(\boldsymbol{e}_2)_2 + \epsilon_{21}(\boldsymbol{e}_1)_2(\boldsymbol{e}_2)_1 \tag{4}$$

$$= \epsilon_{12}\left[(\boldsymbol{e}_1)_1(\boldsymbol{e}_2)_2 - (\boldsymbol{e}_1)_2(\boldsymbol{e}_2)_1\right] \tag{5}$$

$$= \epsilon_{12}[1\times 1 - 0\times 0] \tag{6}$$

$$= \epsilon_{12} \tag{7}$$

where in the third line, we've inserted the components from 2. In order for this volume to be the positive area of the square, we must have

$$\begin{aligned}\epsilon_{12} &= +1\\ \epsilon_{21} &= -1\\ \epsilon_{11} = \epsilon_{22} &= 0\end{aligned} \tag{8}$$

Note that reversing the order of the basis vectors gives

$$\epsilon_{ij}\left(\boldsymbol{e}_2\right)_i\left(\boldsymbol{e}_1\right)_j = \epsilon_{12}\left(\boldsymbol{e}_2\right)_1\left(\boldsymbol{e}_1\right)_2 + \epsilon_{21}\left(\boldsymbol{e}_2\right)_2\left(\boldsymbol{e}_1\right)_1 \tag{9}$$

$$= \epsilon_{12}\left[\left(\boldsymbol{e}_2\right)_1\left(\boldsymbol{e}_1\right)_2 - \left(\boldsymbol{e}_2\right)_2\left(\boldsymbol{e}_1\right)_1\right] \tag{10}$$

$$= \epsilon_{12}\left[0\times 0 - 1\times 1\right] \tag{11}$$

$$= -\epsilon_{12} \tag{12}$$

$$= -1 \tag{13}$$

Ex 1.9(b) By direct substitution, we have

$$\epsilon_{ik}\epsilon_{jk} = \epsilon_{i1}\epsilon_{j1} + \epsilon_{i2}\epsilon_{j2} \tag{14}$$

If $i \neq j$, this expression is zero, since in the first term on the RHS, either $i = 1$ or $j = 1$, which means either $\epsilon_{i1} = \epsilon_{11} = 0$ or $\epsilon_{j1} = \epsilon_{11} = 0$. Similarly, for the second term, either $i = 2$ or $j = 2$, so the term is zero. Thus for a non-zero result, we must have $i = j$. If $i = j = 1$, the first term on the RHS is zero and the second term is $\epsilon_{12}\epsilon_{12} = (+1)(+1) = 1$. If $i = j = 2$, the second term on the RHS is zero and the first term is $\epsilon_{21}\epsilon_{21} = (-1)(-1) = +1$. Thus we have

$$\epsilon_{ik}\epsilon_{jk} = \delta_{ij} \tag{15}$$

PINGBACKS

Pingback: Volume elements in cartesian coordinates