VOLUME ELEMENTS IN CARTESIAN COORDINATES

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References: Kip S. Thorne & Roger D. Blandford, *Modern Classical Physics*, Princeton University Press (2017). Exercise 1.10.

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The volume of a parallelepiped in Euclidean space is given by

$$\epsilon(\boldsymbol{A},\boldsymbol{B},\ldots,\boldsymbol{F}) \equiv \text{volume of paralellepiped} \tag{1}$$

Ex 1.10 In 2-d, the 'volume' element is actually an area element with sides $dx\ \boldsymbol{e}_1$ and $dy\ \boldsymbol{e}_2$. The components of ϵ in 2-d Euclidean space with right-handed orthonormal basis vectors are

$$\begin{aligned} \epsilon_{12} &= +1 \\ \epsilon_{21} &= -1 \\ \epsilon_{11} = \epsilon_{22} &= 0 \end{aligned} \tag{2}$$

Thus the area element is

$$dA = \epsilon(dx\ \boldsymbol{e}_1, dy\ \boldsymbol{e}_2) \tag{3}$$

$$= dx\ dy\ \epsilon(\boldsymbol{e}_1,\boldsymbol{e}_2) \tag{4}$$

$$= dx\ dy\ \epsilon_{ij}(\boldsymbol{e}_1)_i(\boldsymbol{e}_2)_j \tag{5}$$

$$= dx\ dy\ \epsilon_{12} \tag{6}$$

$$= dx\ dy \tag{7}$$

where we used the earlier result

$$\epsilon_{ij}(\boldsymbol{e}_1)_i(\boldsymbol{e}_2)_j = \epsilon_{12} = +1 \tag{8}$$

For a 3-d volume, we use the result given as eqn 1.26 in T&B:

$$\begin{aligned}\epsilon(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C}) &= \epsilon_{ijk}A_iB_jC_k & (9)\\ &= \boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C}) & (10)\\ &= \det\begin{bmatrix} A_1 & B_1 & C_1\\ A_2 & B_2 & C_2\\ A_3 & B_3 & C_3\end{bmatrix} & (11)\end{aligned}$$

For a volume element, we take

$$\begin{aligned}\boldsymbol{A} &= dx\,\boldsymbol{e}_1\\ \boldsymbol{B} &= dy\,\boldsymbol{e}_2\\ \boldsymbol{C} &= dz\,\boldsymbol{e}_3\end{aligned} \qquad (12)$$

Using the product 10 and the orthonormality of the basis vectors, we have

$$\begin{aligned}\boldsymbol{B}\times\boldsymbol{C} &= dy\,dz\,\boldsymbol{e}_2\times\boldsymbol{e}_3 & (13)\\ &= dy\,dz\,\boldsymbol{e}_1 & (14)\\ \boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C}) &= dx\,dy\,dz\,(\boldsymbol{e}_1\cdot\boldsymbol{e}_1) & (15)\\ &= dx\,dy\,dz & (16)\end{aligned}$$

Thus the 3-d volume element in Cartesian coordinates is the usual

$$\text{volume element} = dx\,dy\,dz \qquad (17)$$

We could also derive this result in a more long-winded way by using 11 and the vectors in component form, as

$$\begin{aligned}\boldsymbol{A} &= dx\,\boldsymbol{e}_1 = \begin{bmatrix} dx\\ 0\\ 0\end{bmatrix}\\ \boldsymbol{B} &= dy\,\boldsymbol{e}_2 = \begin{bmatrix} 0\\ dy\\ 0\end{bmatrix}\\ \boldsymbol{C} &= dz\,\boldsymbol{e}_3 = \begin{bmatrix} 0\\ 0\\ dz\end{bmatrix}\end{aligned} \qquad (18)$$

We would then need to write out the components of the determinant, but the end result is the same.

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