INTEGRAL OF A VECTOR FIELD OVER A SPHERE

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References: Kip S. Thorne & Roger D. Blandford, *Modern Classical Physics*, Princeton University Press (2017). Exercise 1.11.

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Suppose we want to integrate the vector field

$$\boldsymbol{A} = z\boldsymbol{e}_z \tag{1}$$

over the surface of a sphere of radius a. That is, we want

$$\int \boldsymbol{A}\cdot d\boldsymbol{\Sigma} \tag{2}$$

where $d\boldsymbol{\Sigma}$ is an outward-pointing surface element on the sphere.

To do this, we introduce spherical coordinates θ,ϕ on the surface of the sphere.

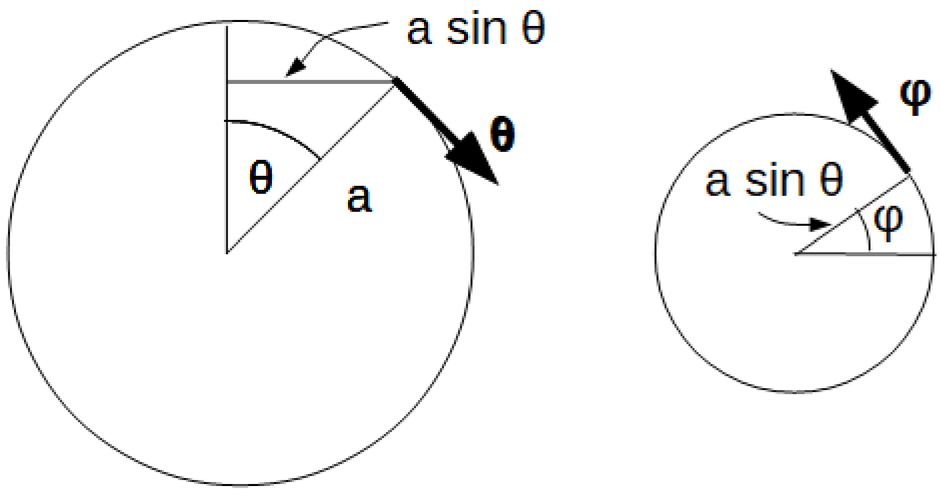

FIGURE 1. Spherical coordinates on a sphere. The LHS shows a vertical cross section through the poles. The RHS shows a horizontal cross section at angle θ.

As usual, the angle θ is the polar angle measured from the north pole (top) of the sphere, and ϕ is the azimuthal angle measured counterclockwise around the sphere (see Fig. 1). A surface element is composed of an infinitesimal displacement in the θ direction times an infinitesimal displacement in the ϕ direction. From the figure, we see that an infinitesimal angle $d\theta$ produces a distance of $a\ d\theta$. The distance produced by $d\phi$ depends on the latitude. On the RHS of the figure, we show a horizontal cross-section of the sphere at angle θ, which has a radius of $a\sin\theta$. Thus a displacement of $d\phi$ produces a distance of $a\sin\theta\ d\phi$ on the surface.

To produce a vectorial surface element $d\mathbf{\Sigma}$, we want a vector that is perpendicular to the surface element and points outwards. If the unit vector in the θ direction is $\boldsymbol{e}_{\hat{\theta}}$ (shown as a bold-face $\boldsymbol{\theta}$ in the figure, as I couldn't figure out how to show $\boldsymbol{e}_{\hat{\theta}}$ in the drawing package) and the unit vector is the ϕ direction is $\boldsymbol{e}_{\hat{\phi}}$ (shown as $\boldsymbol{\phi}$), then the cross product $\boldsymbol{e}_{\hat{\theta}} \times \boldsymbol{e}_{\hat{\phi}}$ is a unit vector that points outwards. Thus we can get the required vector as

$$d\mathbf{\Sigma} = \left(a\ \boldsymbol{e}_{\hat{\theta}}\right) \times \left(a\sin\theta\ \boldsymbol{e}_{\hat{\phi}}\right) d\theta\ d\phi \tag{3}$$

$$= \left(\boldsymbol{e}_{\hat{\theta}} \times \boldsymbol{e}_{\hat{\phi}}\right) a^2 \sin\theta\ d\theta\ d\phi \tag{4}$$

$$= \boldsymbol{\epsilon}\left(\_, \boldsymbol{e}_{\hat{\theta}}, \boldsymbol{e}_{\hat{\phi}}\right) a^2 \sin\theta\ d\theta\ d\phi \tag{5}$$

where $\boldsymbol{\epsilon}\left(\_, \boldsymbol{e}_{\hat{\theta}}, \boldsymbol{e}_{\hat{\phi}}\right)$ is the Levi-Civita tensor, with the first slot unoccupied, so that its components form a vector, with components

$$\boldsymbol{\epsilon}\left(\_, \boldsymbol{e}_{\hat{\theta}}, \boldsymbol{e}_{\hat{\phi}}\right)_i = \left(\boldsymbol{e}_{\hat{\theta}} \times \boldsymbol{e}_{\hat{\phi}}\right)_i \tag{6}$$

To do the integral 2, we need the unit vector $\boldsymbol{e}_z$ in terms of the spherical unit vectors.

From Fig. 2 we see that we can write

$$\boldsymbol{e}_z = \cos\theta \boldsymbol{e}_{\hat{r}} - \sin\theta \boldsymbol{e}_{\hat{\theta}} \tag{7}$$

This follows from the figure, since the projection of $\boldsymbol{e}_z$ on $\boldsymbol{e}_{\hat{r}}$ is $\cos\theta$. The projection of $\boldsymbol{e}_z$ on $\boldsymbol{e}_{\hat{\theta}}$ is the red dashed line in the figure, which is $\sin\theta$, and is in the opposite direction to the vector $\boldsymbol{e}_{\hat{\theta}}$. Thus we get 7.

The quantity z in 1 is the vertical distance measured from the centre of the sphere, so it ranges from $z = -a$ at the south pole to $z = +a$ at the north pole, and is given by

$$z = a\ \cos\theta \tag{8}$$

Since

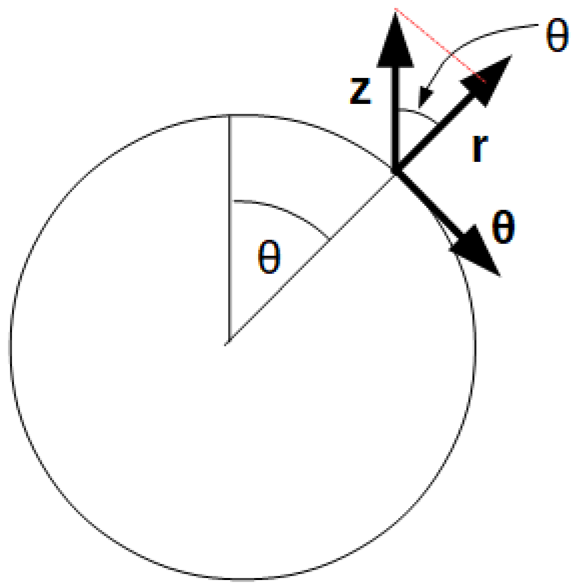

FIGURE 2. Unit vector $\boldsymbol{e}_z$ (labelled bold $\boldsymbol{z}$ in the diagram) in terms of the spherical unit vectors.

$$\boldsymbol{e}_{\hat{\theta}} \times \boldsymbol{e}_{\hat{\phi}} = \boldsymbol{e}_{\hat{r}} \tag{9}$$

and

$$\boldsymbol{e}_{\hat{r}} \cdot \boldsymbol{e}_{\hat{\theta}} = 0 \tag{10}$$

we have

$$\boldsymbol{A} \cdot d\boldsymbol{\Sigma} = a\ \cos\theta \left(\cos\theta \boldsymbol{e}_{\hat{r}} - \sin\theta \boldsymbol{e}_{\hat{\theta}}\right) \cdot \boldsymbol{\epsilon}\left(\_, \boldsymbol{e}_{\hat{\theta}}, \boldsymbol{e}_{\hat{\phi}}\right) a^2 \sin\theta\ d\theta\ d\phi \tag{11}$$

$$= a^2\ \cos\theta \sin\theta \left[\cos\theta \boldsymbol{\epsilon}\left(\boldsymbol{e}_{\hat{r}}, \boldsymbol{e}_{\hat{\theta}}, \boldsymbol{e}_{\hat{\phi}}\right) - \sin\theta \boldsymbol{\epsilon}\left(\boldsymbol{e}_{\hat{\theta}}, \boldsymbol{e}_{\hat{\theta}}, \boldsymbol{e}_{\hat{\phi}}\right)\right]\ d\theta\ d\phi \tag{12}$$

$$= a^3 \cos^2\theta \sin\theta\ d\theta\ d\phi \tag{13}$$

Here we've used the property of the Levi-Civita tensor in 3-d:

$$\epsilon\left(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}\right)\epsilon = \boldsymbol{A} \cdot \left(\boldsymbol{B} \times \boldsymbol{C}\right) \tag{14}$$

The quantity $\boldsymbol{\epsilon}\left(\boldsymbol{e}_{\hat{r}}, \boldsymbol{e}_{\hat{\theta}}, \boldsymbol{e}_{\hat{\phi}}\right) = 1$ because it's the volume of a parallelepiped with 3 orthonormal edges, so its volume is 1. The quantity $\boldsymbol{\epsilon}\left(\boldsymbol{e}_{\hat{\theta}}, \boldsymbol{e}_{\hat{\theta}}, \boldsymbol{e}_{\hat{\phi}}\right) = 0$

because $\boldsymbol{\epsilon}$ is anti-symmetric, so repeating the vector $\boldsymbol{e}_{\hat{\theta}}$ in two slots gives zero.

Integrating 13 over the sphere gives us

$$\int \boldsymbol{A}\cdot d\boldsymbol{\Sigma} = \int_0^{2\pi}\int_0^{\pi} a^3\cos^2\theta\sin\theta\, d\theta\, d\phi \tag{15}$$

$$= 2\pi a^3\left(-\frac{1}{3}\cos^3\theta\right)\Big|_0^{\pi} \tag{16}$$

$$= \frac{4\pi a^3}{3} \tag{17}$$

which is the volume of the sphere.

I'm not sure how to justify this pictorially, but we can observe that applying Gauss's theorem to 2 we have, since $\nabla\cdot\boldsymbol{A} = A_{i,i} = 1$ from 1

$$\int_{\mathcal{S}} \boldsymbol{A}\cdot d\boldsymbol{\Sigma} = \int_{\mathcal{V}} \nabla\cdot\boldsymbol{A}\, d^3x \tag{18}$$

$$= \int_{\mathcal{V}} (1)\, d^3x \tag{19}$$

$$= \frac{4\pi a^3}{3} \tag{20}$$