FARADAY'S LAW OF INDUCTION

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References: Kip S. Thorne & Roger D. Blandford, *Modern Classical Physics*, Princeton University Press (2017). Exercise 1.12.

Post date: 24 Sep 2020.

This exercise asks us to derive Faraday's law of induction from the Maxwell equation

$$\nabla\times\boldsymbol{E}=-\frac{\partial\boldsymbol{B}}{\partial t} \tag{1}$$

This is an inversion of the historical order of events, as Faraday's law was incorporated by Maxwell into his unified theory of electromagnetism. As such, it is an inversion of the derivation we studied earlier.

Starting with 1, we integrate both sites over some closed surface $\mathcal{V}_2$ with a boundary $\partial\mathcal{V}_2$. This gives

$$\int_{\mathcal{V}_2}(\nabla\times\boldsymbol{E})\cdot d\boldsymbol{\Sigma}=-\int_{\mathcal{V}_2}\frac{\partial\boldsymbol{B}}{\partial t}\cdot d\boldsymbol{\Sigma} \tag{2}$$

where $d\boldsymbol{\Sigma}$ is an area element.

We can now apply Stokes's theorem to the LHS to give

$$\int_{\mathcal{V}_2}(\nabla\times\boldsymbol{E})\cdot d\boldsymbol{\Sigma}=\oint_{\partial\mathcal{V}_2}\boldsymbol{E}\cdot d\boldsymbol{\ell} \tag{3}$$

The line integral of the electric field $\boldsymbol{E}$ around a loop is the electromotive force (emf) $\mathcal{E}$ generated in the loop. On the RHS of 2, we can take the derivative outside the integral (since the integral is over space, not time) and we have

$$-\int_{\mathcal{V}_2}\frac{\partial\boldsymbol{B}}{\partial t}\cdot d\boldsymbol{\Sigma}=-\frac{d}{dt}\int_{\mathcal{V}_2}\boldsymbol{B}\cdot d\boldsymbol{\Sigma} \tag{4}$$

The time derivative is now a total derivative, since that is the only variable on which $\int_{\mathcal{V}_2}\boldsymbol{B}\cdot d\boldsymbol{\Sigma}$ depends, as we've integrated over all spatial variables. The integral of magnetic field $\boldsymbol{B}$ over an area is the magnetic flux Φ through that area, so we end up with

$$\oint_{\partial\mathcal{V}_2}\boldsymbol{E}\cdot d\boldsymbol{\ell}=-\frac{d}{dt}\int_{\mathcal{V}_2}\boldsymbol{B}\cdot d\boldsymbol{\Sigma} \tag{5}$$

or

$$\mathcal{E} = -\frac{d\mathbf{\Phi}}{dt} \tag{6}$$

which is Faraday's law of induction. The electromotive force generated in a closed loop is the negative of the rate of change of magnetic flux through the loop.