EQUATIONS OF MOTION FOR A PERFECT FLUID

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References: Kip S. Thorne & Roger D. Blandford, *Modern Classical Physics*, Princeton University Press (2017). Exercise 1.13.

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In Section 1.9, T&B introduce the rank-2 stress tensor T. The tensor is symmetric so that $T_{jk} = T_{kj}$ (as is shown by an analysis of the angular momentum of an element of fluid). The component T_{jk} is the j component of the force per unit area in a direction perpendicular to the basis vector $\boldsymbol{e}_k$. For example, $T_{xy} = T_{12}$ is the component of force F_x in the x direction that acts across an area element perpendicular to $\boldsymbol{e}_y$, that is, an area element parallel to the xz plane.

Since force is rate of change of momentum, we can also interpret T_{jk} to be the j component of momentum that crosses a unit area in a direction perpendicular to the basis vector $\boldsymbol{e}_k$ in unit time.

For a perfect fluid at rest, there are no shear forces so off-diagonal elements of T are zero. If the isotropic pressure in the fluid is P, then $T_{xx} = T_{yy} = T_{zz} = P$, or, in tensor notation, $\mathsf{T} = P\mathsf{g}$ where g is the metric tensor. To see that this works, consider the force in the x direction due to the pressure. The force can be calculated from T by the slot formula

$$\boldsymbol{F}(\_) = \mathsf{T}(\_,\boldsymbol{\Sigma}) \tag{1}$$

$$F_i = T_{ij}\Sigma_j \tag{2}$$

where $\boldsymbol{\Sigma}$ is the vectorial area element. If we take $\boldsymbol{\Sigma}$ to point in the $+x$ direction, then Σ_x is the only non-zero component of $\boldsymbol{\Sigma}$, and the force in the x direction is

$$F_x = T_{xj}\Sigma_j = T_{xx}\Sigma_x \tag{3}$$

If we take $\boldsymbol{\Sigma}$ to point in the $-x$ direction, then its sign is reversed, and the force on the same area element is the negative $-F_x$. This is Newton's law of action-reaction, showing that in a fluid at rest, all forces are balanced.

Ex 1.13(a) We now consider a perfect fluid in motion, and with values of density ρ, pressure P and velocity $\boldsymbol{v}$, all of which can vary in time and space. The momentum $d\boldsymbol{p}$ of an infinitesimal volume dV is its mass times its velocity, so

$$d\boldsymbol{p} = \rho dV \times \boldsymbol{v} \tag{4}$$

The momentum density $\boldsymbol{G}$ is the momentum per unit volume, so

$$\boldsymbol{G} = \frac{d\boldsymbol{p}}{dV} = \rho\boldsymbol{v} \tag{5}$$

To calculate T_{ij} for this fluid, we still have the component due to the pressure mentioned above, so there is still the component Pg. However, since the fluid is in motion, momentum can also be transferred by the motion of the fluid. Consider an infinitesimal volume of fluid that is about to cross an area element $d\boldsymbol{\Sigma}$ with a unit normal vector $\boldsymbol{n}$. If this volume is moving with velocity $\boldsymbol{v}$, the distance it will move across $d\boldsymbol{\Sigma}$ in time dt is $\boldsymbol{v}\cdot\boldsymbol{n}dt$. The volume of fluid that will cross the surface is therefore

$$dV = \boldsymbol{v}\cdot d\boldsymbol{\Sigma}dt \tag{6}$$

so that the mass that crosses the surface is

$$dm = \rho dV = \rho\boldsymbol{v}\cdot d\boldsymbol{\Sigma}dt \tag{7}$$

The momentum of this volume is then

$$d\boldsymbol{p} = dm\,\boldsymbol{v} = (\rho\boldsymbol{v}\cdot d\boldsymbol{\Sigma}dt)\,\boldsymbol{v} \tag{8}$$

Therefore, the j component of momentum that crosses in the k direction per unit area per unit time due to the motion of the fluid is

$$\rho v_j v_k \tag{9}$$

and the total stress tensor is

$$T_{jk} = Pg_{jk} + \rho v_j v_k \tag{10}$$

or, in geometric notation

$$\mathsf{T} = P\mathsf{g} + \rho\boldsymbol{v}\otimes\boldsymbol{v} \tag{11}$$

Ex 1.13(b) To get the law of mass conservation, we consider the rate at which mass flows across the boundary surface. For a surface element $d\boldsymbol{\Sigma}$ consider a volume element that is about to cross this surface with velocity $\boldsymbol{v}$. The volume that crosses the surface in time dt is

$$\text{vol crossing} = \boldsymbol{v}\cdot d\boldsymbol{\Sigma}dt \tag{12}$$

so the mass crossing the surface is

$$\text{mass crossing} = \rho \times (\text{volume crossing}) \tag{13}$$
$$= \rho \boldsymbol{v} \cdot d\boldsymbol{\Sigma} dt \tag{14}$$

so the rate at which mass crosses the surface element is

$$\text{rate} = \rho \boldsymbol{v} \cdot d\boldsymbol{\Sigma} \tag{15}$$

The total rate of change of mass in the volume is the integral of this rate over the entire surface $\partial\mathcal{V}$, so we have

$$\frac{d}{dt}\int_{\mathcal{V}} \rho \, dV = -\int_{\partial\mathcal{V}} \rho \boldsymbol{v} \cdot d\boldsymbol{\Sigma} \tag{16}$$

(negative because if the integral on the RHS is positive, we are losing mass, so the derivative on the LHS is negative).

We can use Gauss's law on the RHS to get

$$\frac{d}{dt}\int_{\mathcal{V}} \rho \, dV = -\int_{\mathcal{V}} \nabla \cdot (\rho \boldsymbol{v}) \, dV \tag{17}$$
$$\int_{\mathcal{V}} \left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) \right) dV = 0 \tag{18}$$

where the full time derivative becomes a partial derivative on the LHS when we take it inside the integral, since ρ is a function of both space and time.

Since this must be true for any volume $\mathcal{V}$, the integrand must be zero, so we have the mass conservation law:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0 \tag{19}$$

Ex 1.13(c) For an observer moving with the fluid, his velocity is $\boldsymbol{v}(\boldsymbol{x}, t)$. The rate of change of a quantity such as density measured by such an observer is composed of two factors. First, the property of the fluid element itself may be changing intrinsically. Second, the property may change because the fluid element has moved to a new location where the values are different. The total derivative can then be worked out using the chain rule. For example, for density ρ, we have

$$\frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + \frac{\partial\rho}{\partial x_i}\frac{\partial x_i}{\partial t} \tag{20}$$

$$= \frac{\partial\rho}{\partial t} + \frac{\partial\rho}{\partial x_i}v_i \tag{21}$$

$$= \frac{\partial\rho}{\partial t} + \boldsymbol{v}\cdot\nabla\rho \tag{22}$$

Thus the total derivative operator is

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla \tag{23}$$

Ex 1.13(d) Returning to 19, we can write it as

$$\frac{\partial\rho}{\partial t} = -\nabla\cdot(\rho\boldsymbol{v}) \tag{24}$$

$$= -\boldsymbol{v}\cdot\nabla\rho - \rho\nabla\cdot\boldsymbol{v} \tag{25}$$

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla\right)\rho = -\rho\nabla\cdot\boldsymbol{v} \tag{26}$$

$$\frac{d\rho}{dt} = -\rho\nabla\cdot\boldsymbol{v} \tag{27}$$

$$\frac{1}{\rho}\frac{d\rho}{dt} = -\nabla\cdot\boldsymbol{v} \tag{28}$$

Ex 1.13(e) The differential law of momentum conservation given in T&B is their eqn 1.36, in both geometric and index forms:

$$\frac{\partial\boldsymbol{G}}{\partial t} + \nabla\cdot\mathsf{T} = 0 \tag{29}$$

$$\frac{\partial G_j}{\partial t} + T_{jk;k} = 0 \tag{30}$$

To apply this to 5 and 11, it's probably easier to work with components initially. First, we have

$$\frac{\partial G_j}{\partial t} = v_j\frac{\partial\rho}{\partial t} + \rho\frac{\partial v_j}{\partial t} \tag{31}$$

or

$$\frac{\partial\boldsymbol{G}}{\partial t} = \frac{\partial\rho}{\partial t}\boldsymbol{v} + \rho\frac{\partial\boldsymbol{v}}{\partial t} \tag{32}$$

We need to find $\nabla\cdot(\rho\boldsymbol{v}\otimes\boldsymbol{v})$. In components, we have

$$\nabla\cdot(\rho\boldsymbol{v}\otimes\boldsymbol{v})_j=\rho_{;k}v_jv_k+\rho v_{j;k}v_k+\rho v_jv_{k;k} \tag{33}$$

Converting back to geometric notation we have

$$\nabla\cdot(\rho\boldsymbol{v}\otimes\boldsymbol{v})=(\boldsymbol{v}\cdot\nabla\rho)\,\boldsymbol{v}+\rho\,(\boldsymbol{v}\cdot\nabla)\,\boldsymbol{v}+\rho\boldsymbol{v}\,(\nabla\cdot\boldsymbol{v}) \tag{34}$$

We also have, since $g_{jk}=\delta_{jk}$

$$\nabla\cdot(P\mathrm{g})_j=P_{;k}g_{jk}=P_{;j} \tag{35}$$

which is

$$\nabla\cdot(P\mathrm{g})=\nabla P \tag{36}$$

Combining 36 and 34 we have

$$\nabla\cdot\mathsf{T}=\nabla P+(\boldsymbol{v}\cdot\nabla\rho)\,\boldsymbol{v}+\rho\,(\boldsymbol{v}\cdot\nabla)\,\boldsymbol{v}+\rho\boldsymbol{v}\,(\nabla\cdot\boldsymbol{v}) \tag{37}$$

so from 37 and 32 the momentum conservation law 29 becomes

$$\frac{\partial\rho}{\partial t}\boldsymbol{v}+\rho\frac{\partial\boldsymbol{v}}{\partial t}+\nabla P+(\boldsymbol{v}\cdot\nabla\rho)\,\boldsymbol{v}+\rho\,(\boldsymbol{v}\cdot\nabla)\,\boldsymbol{v}+\rho\boldsymbol{v}\,(\nabla\cdot\boldsymbol{v})=0 \tag{38}$$

Using 23, we can convert this to

$$\boldsymbol{v}\left(\frac{\partial\rho}{\partial t}+\boldsymbol{v}\cdot\nabla\rho\right)+\rho\left(\frac{\partial\boldsymbol{v}}{\partial t}+(\boldsymbol{v}\cdot\nabla)\,\boldsymbol{v}\right)+\nabla P+\rho\boldsymbol{v}\,(\nabla\cdot\boldsymbol{v})=0 \tag{39}$$

$$\boldsymbol{v}\frac{d\rho}{dt}+\rho\frac{d\boldsymbol{v}}{dt}+\nabla P-\boldsymbol{v}\frac{d\rho}{dt}=0 \tag{40}$$

$$\rho\frac{d\boldsymbol{v}}{dt}+\nabla P=0 \tag{41}$$

where we used 28 to get the last term in the second line. We thus have the result:

$$\frac{d\boldsymbol{v}}{dt}=-\frac{\nabla P}{\rho} \tag{42}$$

Since $\frac{d\boldsymbol{v}}{dt}$ is the acceleration $\boldsymbol{a}$ of the fluid at each point, this is essentially Newton's law $\boldsymbol{F}=m\boldsymbol{a}$ where the LHS is the force per unit mass.

PINGBACKS

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