ELECTROMAGNETIC STRESS TENSOR

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

References: Kip S. Thorne & Roger D. Blandford, *Modern Classical Physics*, Princeton University Press (2017). Exercise 1.14.

Post date: 28 Sep 2020.

I'll start with a disclaimer that this isn't really a solution to part (a) of this exercise, as I couldn't figure out how to do it in the manner required.

The idea is to start with the facts that an electric field exerts a pressure of magnitude $\epsilon_0 \boldsymbol{E}^2/2$ orthogonal to itself and a tension of the same magnitude along itself. A magnetic field also exerts a pressure of $\epsilon_0 c^2 \boldsymbol{B}^2/2$ orthogonal to itself and a tension of the same magnitude along itself. We are then asked to verify that the given stress tensor embodies these stresses. The stress tensor is

$$\mathsf{T} = \frac{\epsilon_0}{2}\left[\left(\boldsymbol{E}^2 + c^2\boldsymbol{B}^2\right)\mathsf{g} - 2\left(\boldsymbol{E}\otimes\boldsymbol{E} + c^2\boldsymbol{B}\otimes\boldsymbol{B}\right)\right] \tag{1}$$

where g is the metric tensor, with $g_{ij} = \delta_{ij}$.

We can see that this works for some special cases. For example, suppose we take the magnetic field to be $\boldsymbol{B} = 0$, and the electric field to be along the x axis, so that

$$\boldsymbol{E} = [E, 0, 0] \tag{2}$$

Then 1 gives us

$$T_{xx} = \frac{\epsilon_0}{2}\left(E^2 - 2E^2\right) \tag{3}$$

$$= -\frac{\epsilon_0}{2}E^2 \tag{4}$$

$$T_{yy} = T_{zz} = \frac{\epsilon_0}{2}E^2 \tag{5}$$

with all off-diagonal elements equal to zero.

Now we consider a surface element $\boldsymbol{\Sigma}$ normal to the x axis, so that

$$\boldsymbol{\Sigma} = [A, 0, 0] \tag{6}$$

Then the force on this area element is given by

$$\begin{aligned} F_x &= T_{xj}\Sigma_j && (7) \\ &= T_{xx}A && (8) \\ &= -\frac{\epsilon_0}{2}E^2A && (9) \\ F_y &= F_z = 0 && (10) \end{aligned}$$

This does indeed correspond to a tension force along the x axis. The components of force orthogonal to $\boldsymbol{\Sigma}$ are zero, since the pressure exerted by the field is orthogonal to $\boldsymbol{\Sigma}$ and so exerts no force on the surface element.

If we take a surface element normal to the y axis, so that

$$\boldsymbol{\Sigma} = [0, A, 0] \tag{11}$$

then we get

$$\begin{aligned} F_x &= F_z = 0 && (12) \\ F_y &= \frac{\epsilon_0}{2}E^2A && (13) \end{aligned}$$

In this case, the component F_y arises from the pressure so is positive. The tension is orthogonal to $\boldsymbol{\Sigma}$ so exerts no force on the area element.

If we now consider an electric field in the xy plane, so that

$$\boldsymbol{E} = [E_x, E_y, 0] \tag{14}$$

then the stress tensor becomes

$$\begin{aligned} T_{xx} &= \frac{\epsilon_0}{2}\left(-E_x^2 + E_y^2\right) && (15) \\ T_{yy} &= \frac{\epsilon_0}{2}\left(E_x^2 - E_y^2\right) && (16) \\ T_{zz} &= \frac{\epsilon_0}{2}\left(E_x^2 + E_y^2\right) && (17) \\ T_{xy} &= T_{yx} = -\epsilon_0 E_x E_y && (18) \end{aligned}$$

with the remaining elements equal to zero.

For an area element orthogonal to the x axis, the force is

$$F_x = T_{xj}\Sigma_j = \frac{\epsilon_0}{2}\left(-E_x^2 + E_y^2\right)A \tag{19}$$

We see there is a negative tension force from E_x and a positive pressure force from E_y, which satisfy the requirements above. However, there is now a y component to the force:

$$F_y = T_{yj}\Sigma_j \quad (20)$$
$$= T_{yx}A \quad (21)$$
$$= -\epsilon_0 E_x E_y A \quad (22)$$

According to the standard interpretation of the stress tensor, off-diagonal elements such as T_{yx} represent shear forces, so there is a shear force in this case. Presumably this comes from the tension force, but I can't see any way of justifying this. Comments welcome.

In any case, we previously derived the stress tensor by considering the force on a collection of charges and currents. This derivation starts with the Lorentz force law in the form

$$\boldsymbol{F} = \int_{\mathcal{V}} \rho\left(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B}\right)d\mathcal{V} \quad (23)$$
$$= \int_{\mathcal{V}} \left(\rho\boldsymbol{E}+\boldsymbol{J}\times\boldsymbol{B}\right)d\mathcal{V} \quad (24)$$

and uses Maxwell's equations to derive the stress tensor. This derivation makes no mention of pressure or tension so isn't technically a solution to the exercise here, but at least we can see how to derive the stress tensor.

Part (b) of the exercise asks us to take the divergence of T and show that it produces the force on a collection of charges and currents, using Maxwell's equations. This is effectively a reversal of the steps taken in the previous post. In that post, we start with equation (23) to calculate $\nabla\cdot\mathsf{T}$ and then compare this with equation (20) which shows the force density $\boldsymbol{f}$, which is the integrand of 24:

$$\boldsymbol{f} = \rho\boldsymbol{E}+\boldsymbol{J}\times\boldsymbol{B} \quad (25)$$

The result, for the case of steady fields that don't change with time, is

$$\boldsymbol{f} = -\nabla\cdot\mathsf{T} \quad (26)$$

(eqn (26) in the previous post). The minus sign here is due to the fact that the tensor T defined by T&B 1 is the negative of the one used by Griffiths.