ANTISYMMETRY OF THE ELECTROMAGNETIC FIELD TENSOR

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References: Kip S. Thorne & Roger D. Blandford, *Modern Classical Physics*, Princeton University Press (2017). Exercise 2.3.

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In their eqn 2.17, T&B show that the four-momentum $\vec{p}$ must always be orthogonal to the four-force $\vec{F}$ in the sense that

$$\vec{p}\cdot\vec{F}=0 \tag{1}$$

Note that this a 4-dimensional inner product and uses relativistic flat-space metric, so it's equivalent to (in components)

$$\eta_{jk}p^jF^k=0 \tag{2}$$

In section 2.4.2, they derive the form of the electromagnetic field tensor, and show that it is a rank-2 tensor F with the property

$$\mathsf{F}(\vec{u},\vec{u})=0 \tag{3}$$

where $\vec{u}$ is the timelike 4-velocity of the test charge.

Ex. 2.3. We are to show that 3 is true if and only if F is antisymmetric. The 'if' part is trivial, since if F is antisymmetric, then swapping the two vectors in its arguments gives the negative of the original, so if the two vectors are the same, then we have

$$\mathsf{F}(\vec{u},\vec{u})=-\mathsf{F}(\vec{u},\vec{u})=0 \tag{4}$$

We therefore need to show that if 3 is true, then F must be antisymmetric.

First, we can write a general rank-2 tensor as the sum of a symmetric tensor S and an antisymmetric tensor A. We have

$$\mathsf{F}\left(\vec{a},\vec{b}\right)=\mathsf{A}\left(\vec{a},\vec{b}\right)+\mathsf{S}\left(\vec{a},\vec{b}\right) \tag{5}$$

Swapping the vectors in the argument and using the symmetry and antisymmetry gives us

$$\mathsf{F}\left(\vec{b},\vec{a}\right) = \mathsf{A}\left(\vec{b},\vec{a}\right) + \mathsf{S}\left(\vec{b},\vec{a}\right) \tag{6}$$

$$= -\mathsf{A}\left(\vec{a},\vec{b}\right) + \mathsf{S}\left(\vec{a},\vec{b}\right) \tag{7}$$

Combining 5 and 7 we have

$$\mathsf{A}\left(\vec{a},\vec{b}\right) = \frac{1}{2}\left(\mathsf{F}\left(\vec{a},\vec{b}\right) - \mathsf{F}\left(\vec{b},\vec{a}\right)\right) \tag{8}$$

$$\mathsf{S}\left(\vec{a},\vec{b}\right) = \frac{1}{2}\left(\mathsf{F}\left(\vec{a},\vec{b}\right) + \mathsf{F}\left(\vec{b},\vec{a}\right)\right) \tag{9}$$

Thus we can always decompose a general tensor F into a symmetric and antisymmetric part. The antisymmetric part satisfies

$$\mathsf{A}\left(\vec{u},\vec{u}\right) = 0 \tag{10}$$

for the reason given above. We know from 3 and 5 that $\mathsf{S}\left(\vec{u},\vec{u}\right) = 0$ for timelike $\vec{u}$. We would like to show that

We want to show that 3 is true for all timelike vectors $\vec{u}$. Let $\vec{a}$ be a timelike vector and $\vec{b}$ be an arbitrary (not necessarily timelike) vector. We define

$$\vec{A}_{\pm} \equiv \vec{a} \pm \epsilon\vec{b} \tag{11}$$

where ϵ is small enough that both $\vec{A}_{\pm}$ are still timelike. Using the linearity of tensors, we have

$$\mathsf{S}\left(\vec{A}_{+},\vec{A}_{+}\right) = \mathsf{S}\left(\vec{a}+\epsilon\vec{b},\vec{a}+\epsilon\vec{b}\right) \tag{12}$$

$$= \mathsf{S}\left(\vec{a}+\epsilon\vec{b},\vec{a}\right) + \epsilon\mathsf{S}\left(\vec{a}+\epsilon\vec{b},\vec{b}\right) \tag{13}$$

$$= \mathsf{S}\left(\vec{a},\vec{a}\right) + \epsilon\mathsf{S}\left(\vec{b},\vec{a}\right) + \epsilon\mathsf{S}\left(\vec{a},\vec{b}\right) + \epsilon^2\mathsf{S}\left(\vec{b},\vec{b}\right) \tag{14}$$

$$= \mathsf{S}\left(\vec{a},\vec{a}\right) + 2\epsilon\mathsf{S}\left(\vec{a},\vec{b}\right) + \epsilon^2\mathsf{S}\left(\vec{b},\vec{b}\right) \tag{15}$$

where the last line follows because S is symmetric, so

$$\mathsf{S}\left(\vec{b},\vec{a}\right) = \mathsf{S}\left(\vec{a},\vec{b}\right) \tag{16}$$

Similarly, we have

$$\mathsf{S}\left(\vec{A}_-,\vec{A}_-\right) = \mathsf{S}\left(\vec{a}-\epsilon\vec{b},\vec{a}-\epsilon\vec{b}\right) \tag{17}$$
$$= \mathsf{S}\left(\vec{a}-\epsilon\vec{b},\vec{a}\right) - \epsilon\mathsf{S}\left(\vec{a}-\epsilon\vec{b},\vec{b}\right) \tag{18}$$
$$= \mathsf{S}\left(\vec{a},\vec{a}\right) - \epsilon\mathsf{S}\left(\vec{b},\vec{a}\right) - \epsilon\mathsf{S}\left(\vec{a},\vec{b}\right) + \epsilon^2\mathsf{S}\left(\vec{b},\vec{b}\right) \tag{19}$$
$$= \mathsf{S}\left(\vec{a},\vec{a}\right) - 2\epsilon\mathsf{S}\left(\vec{a},\vec{b}\right) + \epsilon^2\mathsf{S}\left(\vec{b},\vec{b}\right) \tag{20}$$

Since we require $\mathsf{S}\left(\vec{u},\vec{u}\right)=0$ for all timelike vectors $\vec{u}$, and $\vec{A}_\pm$ and $\vec{a}$ are all timelike vectors, then we have

$$\mathsf{S}\left(\vec{A}_+,\vec{A}_+\right) = \mathsf{S}\left(\vec{A}_-,\vec{A}_-\right) = \mathsf{S}\left(\vec{a},\vec{a}\right) = 0 \tag{21}$$

Thus 15 and 20 become

$$2\mathsf{S}\left(\vec{a},\vec{b}\right) + \epsilon\mathsf{S}\left(\vec{b},\vec{b}\right) = 0 \tag{22}$$
$$-2\mathsf{S}\left(\vec{a},\vec{b}\right) + \epsilon\mathsf{S}\left(\vec{b},\vec{b}\right) = 0 \tag{23}$$

Adding these two equations gives us

$$\mathsf{S}\left(\vec{b},\vec{b}\right) = 0 \tag{24}$$

for *all* (not just timelike) vectors $\vec{b}$. Plugging this back into 22 gives us

$$\mathsf{S}\left(\vec{a},\vec{b}\right) = 0 \tag{25}$$

This shows that S is zero when one vector is timelike and the other is arbitrary. To show that S is still zero if neither vector is timelike, we can repeat the derivation above, but this time we start with a vector $\vec{c}$ that is *not* timelike and define

$$\vec{C}_\pm \equiv \vec{c} \pm \epsilon\vec{d} \tag{26}$$

where $\vec{d}$ is arbitrary, and ϵ is small enough that both $\vec{C}_\pm$ are not timelike. Since we know from 24 that S is zero whenever its two arguments are the same, we can apply this to $\mathsf{S}\left(\vec{C}_\pm,\vec{C}_\pm\right)$, $\mathsf{S}\left(\vec{c},\vec{c}\right)$ and $\mathsf{S}\left(\vec{d},\vec{d}\right)$ to show that

$$\mathsf{S}\left(\vec{c},\vec{d}\right) = 0 \tag{27}$$

That is, S is zero when one of its arguments is not timelike and the other is arbitrary. This, combined with 25 shows that S is zero for any two vectors as its arguments, and thus

$$\mathsf{F}\left(\vec{A},\vec{B}\right)=\mathsf{A}\left(\vec{A},\vec{B}\right)=-\mathsf{A}\left(\vec{B},\vec{A}\right) \tag{28}$$

so that F is antisymmetric. As we've seen earlier, the explicit form of the tensor in components is

$$F^{ij}=\left[\begin{array}{cccc} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{array}\right] \tag{29}$$

and is indeed antisymmetric.