RELATIVISTIC COMPONENT MANIPULATION RULES

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References: Kip S. Thorne & Roger D. Blandford, *Modern Classical Physics*, Princeton University Press (2017). Exercise 2.5.

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We've seen the tensor component manipulation rules in 3-d Euclidean space. In that case, a vector $\boldsymbol{A}$ can be written in components as

$$\boldsymbol{A} = A_j \boldsymbol{e}_j \tag{1}$$

where $\boldsymbol{e}_j$ are the orthonormal basis vectors that obey

$$\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij} \tag{2}$$

A general tensor T has components given by

$$\mathsf{T} = T_{ijk} \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \tag{3}$$

In Minkowski spacetime, we have 4 basis vectors $\vec{e}_\alpha$ which obey the rule

$$\vec{e}_\alpha \cdot \vec{e}_\beta = \eta_{\alpha\beta} \tag{4}$$

where

$$\begin{aligned} \eta_{00} &\equiv -1 \\ \eta_{ij} &\equiv \delta_{ij} \end{aligned} \tag{5}$$

A basis in which 4 holds is said to be orthonormal in Minkowski spacetime. Note that the orthogonality implies that $\vec{e}_0 \cdot \vec{e}_0 = -1$.

Unlike 3-d Euclidean space, the behaviour of the 0 component in 4-d spacetime requires introducing two types of components for tensors. The *contravariant* components are defined in analogy to 1, and are written with raised indices, so we have

$$\vec{A} = A^\alpha \vec{e}_\alpha \tag{6}$$

That is, a vector $\vec{A}$ is expanded in terms of the basis vectors by using its contravariant components. As usual, a repeated index (α) implies a sum, but in 4-d spacetime, a pair of repeated indices must include one upper and one lower index. In 4-d spacetime, the sum is over 0,1,2,3. In 3-d space, all indices were lower, and a pair of repeated indices is summed over 1,2,3.

To get the lower, or *covariant* components of a vector, we insert a basis vector into the vector's slot, so we have

$$A_\beta = \vec{A}\left(\vec{e}_\beta\right) \tag{7}$$
$$= A^\alpha \vec{e}_\alpha \cdot \vec{e}_\beta \tag{8}$$
$$= A^\alpha \eta_{\alpha\beta} \tag{9}$$

We therefore have the rule that lowering the 0 index changes the sign of the component, while lowering any of the indices 1,2,3 leaves the component unchanged.

For a general tensor, the analog of 3 is

$$\mathsf{T} = T^{\alpha\beta\gamma} \vec{e}_\alpha \otimes \vec{e}_\beta \otimes \vec{e}_\gamma \tag{10}$$

The lowered components are obtained by inserting the required basis vectors into the slots of T, so we have

$$T_{\delta\epsilon\zeta} = \mathsf{T}\left(\vec{e}_\delta, \vec{e}_\epsilon, \vec{e}_\zeta\right) \tag{11}$$
$$= T^{\alpha\beta\gamma}\left(\vec{e}_\alpha \cdot \vec{e}_\delta\right)\left(\vec{e}_\beta \cdot \vec{e}_\epsilon\right)\left(\vec{e}_\gamma \cdot \vec{e}_\zeta\right) \tag{12}$$
$$= T^{\alpha\beta\gamma} \eta_{\alpha\delta} \eta_{\beta\epsilon} \eta_{\gamma\zeta} \tag{13}$$

Thus lowering any 0 index changes the sign, but lowering any other index doesn't change the sign.

The metric tensor g is a rank-2 tensor defined so that inserting two vectors into its slots gives the inner product:

$$\mathsf{g}\left(\vec{A}, \vec{B}\right) \equiv \vec{A} \cdot \vec{B} \tag{14}$$

Thus the components of g are, from 4

$$g_{\alpha\beta} = \mathsf{g}\left(\vec{e}_\alpha, \vec{e}_\beta\right) = \vec{e}_\alpha \cdot \vec{e}_\beta = \eta_{\alpha\beta} \tag{15}$$

The metric tensor is also written in the form 10 as

$$\mathsf{g} = g^{\alpha\beta} \vec{e}_\alpha \otimes \vec{e}_\beta \tag{16}$$

Comparing this with 15 we have

$$g_{\alpha\beta} = \mathsf{g}\left(\vec{e}_\alpha, \vec{e}_\beta\right) \tag{17}$$
$$= g^{\gamma\delta}\left(\vec{e}_\gamma \cdot \vec{e}_\alpha\right)\left(\vec{e}_\delta \cdot \vec{e}_\beta\right) \tag{18}$$
$$= g^{\gamma\delta} \eta_{\gamma\alpha} \eta_{\delta\beta} \tag{19}$$

Since η has the values 5, we see that

$$g_{\alpha\beta} = g^{\alpha\beta} \tag{20}$$

for all components.

Thus we get the standard rule for raising or lowering tensor indices by multiplying by the metric.

Ex. 2.5 We can now derive a few relations. First, we want the contravariant components of $\mathsf{T}(\_,\_,\_) \otimes \mathsf{S}(\_,\_)$. From 10 we have

$$\mathsf{T} \otimes \mathsf{S} = \left[T^{\alpha\beta\gamma}\vec{e}_\alpha \otimes \vec{e}_\beta \otimes \vec{e}_\gamma\right] \otimes \left[S^{\delta\epsilon}\vec{e}_\delta \otimes \vec{e}_\epsilon\right] \tag{21}$$

$$= T^{\alpha\beta\gamma}S^{\delta\epsilon}\vec{e}_\alpha \otimes \vec{e}_\beta \otimes \vec{e}_\gamma \otimes \vec{e}_\delta \otimes \vec{e}_\epsilon \tag{22}$$

Thus $\mathsf{T} \otimes \mathsf{S}$ is a rank-5 tensor with contravariant components $T^{\alpha\beta\gamma}S^{\delta\epsilon}$.

From 10 we have

$$\mathsf{T}\left(\vec{A},\vec{B},\vec{C}\right) = T^{\alpha\beta\gamma}\left(\vec{A}\cdot\vec{e}_\alpha\right)\left(\vec{B}\cdot\vec{e}_\beta\right)\left(\vec{C}\cdot\vec{e}_\gamma\right) \tag{23}$$

From 6 and 9 we have

$$\vec{A}\cdot\vec{e}_\alpha = A^\beta\vec{e}_\beta\cdot\vec{e}_\alpha \tag{24}$$

$$= A^\beta\eta_{\beta\alpha} \tag{25}$$

$$= A_\alpha \tag{26}$$

Therefore

$$\mathsf{T}\left(\vec{A},\vec{B},\vec{C}\right) = T^{\alpha\beta\gamma}A_\alpha B_\beta C_\gamma \tag{27}$$

By applying the raising and lowering rules as in 13, we can lower the indices on $T^{\alpha\beta\gamma}$ and raise those on $A_\alpha B_\beta C_\gamma$ to get

$$\mathsf{T}\left(\vec{A},\vec{B},\vec{C}\right) = T_{\alpha\beta\gamma}A^\alpha B^\beta C^\gamma \tag{28}$$

From 14 and 16 we have

$$\vec{A}\cdot\vec{B} = \mathsf{g}\left(\vec{A},\vec{B}\right) \tag{29}$$

$$= g_{\alpha\beta}A^\alpha B^\beta \tag{30}$$

$$= A_\beta B^\beta \tag{31}$$

$$= A^\alpha B_\alpha \tag{32}$$

where to get the third and fourth lines, we applied $g_{\alpha\beta}$ first to lower the index on A^{α} and then, alternatively, to lower the index on B^{β}.

We now consider a rank-4 tensor R. To contract on the 1st and 3rd slots, we need one of these slots to be upper and the other lower, so if we start with

$$R^{\gamma\alpha\delta\beta} \tag{33}$$

we then need to lower, say, the third component by multiplying by the metric, so we have

$$g_{\delta\zeta}R^{\gamma\alpha\zeta\beta}=R^{\gamma\alpha}{}_{\delta}{}^{\beta} \tag{34}$$

We can now contract the γ and δ indices by setting $\gamma=\delta$ and summing, so we have

$$g_{\gamma\zeta}R^{\gamma\alpha\zeta\beta}=R^{\gamma\alpha}{}_{\gamma}{}^{\beta} \tag{35}$$

Since the uncontracted indices α and β are upper, the result is the contravariant component form of the contraction. We could have lowered the γ index in 33 instead of the δ index, and the resulting contraction would then be $R_{\gamma}{}^{\alpha\gamma\beta}$ which would be the same as $R^{\gamma\alpha}{}_{\gamma}{}^{\beta}$.

To get the covariant form, we apply the metric to each of the indices α and β to lower them, with the result:

$$R^{\gamma}{}_{\alpha\gamma\beta}=R_{\gamma\alpha}{}^{\gamma}{}_{\beta} \tag{36}$$

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