SLOT-NAMING INDEX NOTATION AND INDEX GYMNASTICS

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References: Kip S. Thorne & Roger D. Blandford, *Modern Classical Physics*, Princeton University Press (2017). Exercises 2.7-2.8.

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A few examples of the relation between tensor index notation and the geometric index-free notation.

Ex 2.7 (a). As far as I can tell (from looking at T&B's eqns 2.23, for example), there is no distinction between contravariant and covariant indices when writing a tensor expression in geometric form. In their equations 2.23e and 2.23g, for example, they write out explicitly whether indices are contravariant or covariant. In that case, both $A^{\alpha}B_{\gamma\delta}$ and $A_{\alpha}B_{\gamma}{}^{\delta}$ would be written as $\vec{A}(\_)\otimes\mathsf{B}(\_,\_)$. We need to introduce a basis to distinguish between contravariant and covariant indices.

The equation

$$S_{\alpha}{}^{\beta\gamma}=S^{\gamma\beta}{}_{\alpha} \tag{1}$$

indicates that the tensor is symmetric when swapping its first and third slots, so we have

$$\mathsf{S}\left(\vec{A},\vec{B},\vec{C}\right)=\mathsf{S}\left(\vec{C},\vec{B},\vec{A}\right) \tag{2}$$

The equation

$$A^{\alpha}B_{\alpha}=A_{\alpha}B^{\beta}g^{\alpha}{}_{\beta} \tag{3}$$

would be written as

$$\vec{A}\left(\vec{B}\right)=\vec{A}\left(\mathsf{g}\left(\_,\vec{B}\right)\right) \tag{4}$$

or, equivalently

$$\vec{A}\left(\vec{B}\right)=\mathsf{g}\left(\vec{A},\vec{B}\right)=\vec{B}\left(\vec{A}\right) \tag{5}$$

Ex 2.7 (b). We now want to convert to index notation the geometric expression

$$\mathsf{T}\left(\_,\mathsf{S}\left(\mathsf{R}\left(\vec{C},\_\right),\_\right),\_\right) \tag{6}$$

we start at the inside and work outwards. We have

$$\mathsf{R}\left(\vec{C},\_\right) \to R^{\alpha\beta}C_{\alpha} \tag{7}$$

I'm using an arrow $\to$ instead of $=$ since in the following, the object on the LHS is a full tensor while the object on the RHS is just one component of this tensor.

This object is a vector with the free index β. This vector is inserted into the first slot of S so we have

$$\mathsf{S}\left(\mathsf{R}\left(\vec{C},\_\right),\_\right) \to S_{\beta\gamma}R^{\alpha\beta}C_{\alpha} \tag{8}$$

This object is also a vector, as it has one free index γ. This is inserted into the second slot of T, so we have

$$\mathsf{T}\left(\_,\mathsf{S}\left(\mathsf{R}\left(\vec{C},\_\right),\_\right),\_\right) \to T^{\delta\gamma\epsilon}S_{\beta\gamma}R^{\alpha\beta}C_{\alpha} \tag{9}$$

The final object is a rank-2 tensor with free indices δ and ϵ. In this form, we have given its contravariant components since both δ and ϵ are upper indices. Note that we had to choose one upper and one lower index for each pair of summed indices, which is why α, β and γ appear in the positions they do. We could equally have written this result as

$$\mathsf{T}\left(\_,\mathsf{S}\left(\mathsf{R}\left(\vec{C},\_\right),\_\right),\_\right) \to T_{\delta\ \ \epsilon}^{\ \ \gamma}S_{\beta\gamma}R^{\alpha\beta}C_{\alpha} \tag{10}$$

to get the covariant components. We could also swap positions of any pair of summed indices, as in

$$\mathsf{T}\left(\_,\mathsf{S}\left(\mathsf{R}\left(\vec{C},\_\right),\_\right),\_\right) \to T^{\delta\ \ \epsilon}_{\ \ \gamma}S^{\beta\gamma}R^{\alpha}_{\ \ \beta}C_{\alpha} \tag{11}$$

or any other combination that obeys these rules.

Ex 2.8 (a) Since the metric tensor is used to raise or lower indices in other tensors, we can simplify

$$A^{\alpha\beta\gamma}g_{\beta\rho}S_{\gamma\lambda}g^{\rho\delta}g^{\lambda}_{\ \ \alpha} \tag{12}$$

First, we have

$$g_{\beta\rho}g^{\rho\delta} = g_{\beta}^{\ \ \delta} = \delta_{\beta\delta} \tag{13}$$

where the last equality follows from T&B's eqn 2.23c. Also

$$g^{\lambda}_{\ \ \alpha} = \delta_{\lambda\alpha} \tag{14}$$

so we set $\delta = \beta$ and $\lambda = \alpha$ in 12 to get

$$A^{\alpha\beta\gamma}g_{\beta\rho}S_{\gamma\lambda}g^{\rho\delta}g^{\lambda}{}_{\alpha} = A^{\alpha\beta\gamma}S_{\gamma\lambda}g_{\beta}{}^{\delta}\delta_{\lambda\alpha} \tag{15}$$

$$= A^{\alpha\delta\gamma}S_{\gamma\alpha} \tag{16}$$

Since α and γ are summed, this final form is a vector with one free index δ.

Ex. 2.8 (b) The double contraction on the metric tensor gives us

$$g_{\alpha\beta}g^{\alpha\beta} = g_{\alpha}{}^{\alpha} \tag{17}$$

$$= \sum_{\alpha=0}^{3}\delta_{\alpha\alpha} \tag{18}$$

$$= 4 \tag{19}$$

In the first line, we used the $g^{\alpha\beta}$ to raise the β index on the $g_{\alpha\beta}$, which gives the sum over the Kronecker delta in the second line.

Ex. 2.8 (c) The expression

$$A_{\alpha}{}^{\beta\gamma}S_{\alpha\gamma} \tag{20}$$

is invalid, since both α indices appear in the lower position, which violates the rule that a summed pair must consist of one lower and one upper index.

The equation

$$A_{\alpha}{}^{\beta\gamma}S_{\beta}T_{\gamma} = R_{\alpha\beta\delta}S^{\beta} \tag{21}$$

is invalid since the LHS contains only one free index α while the RHS contains 2: α and δ. It's also a bit confusing in that the same index β is used for an implied sum on both sides of the equation, in different contexts.