# 3-METRIC AS A PROJECTION TENSOR

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Reference: Kip S. Thorne & Roger D. Blandford, *Modern Classical Physics*, Princeton University Press (2017). Exercise 2.10.

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We consider an observer with 4-velocity $\vec{U}$ who measures the properties of a particle with 4-momentum $\vec{p}$. Consider first the tensor

$$\mathsf{P}\equiv\mathsf{g}+\vec{U}\otimes\vec{U} \tag{1}$$

In the observer's frame, his 4-velocity is

$$\vec{U}=(1,0,0,0) \tag{2}$$

so in that frame, the components of $\mathsf{P}$ are

$$P^{00}=g^{00}+U^{0}U^{0} \tag{3}$$
$$=-1+1 \tag{4}$$
$$=0 \tag{5}$$
$$P^{kk}=g^{kk}+U^{k}U^{k} \tag{6}$$
$$=1+0 \tag{7}$$
$$=1 \tag{8}$$

where there is no sum over $k$ in the expression for $P^{kk}$. All off-diagonal elements $P^{\alpha\beta}$ are zero. Thus 1 is the Euclidean metric of the observer's 3-space, regarded as a tensor in 4-dimensional spacetime.

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If $\vec{A}$ is an arbitrary vector in spacetime, then its component along $\vec{U}$ is, in the observer's frame, the component along the $\vec{e}_0$ basis vector. This component is

$$A^{0}=-A_{0} \tag{9}$$
$$=-\vec{A}\cdot\vec{e}_{0} \tag{10}$$
$$=-\vec{A}\cdot\vec{U} \tag{11}$$

The object $-\vec{A}\cdot\vec{U}$ is a geometric scalar (no reference to coordinate system) and thus has the same value in all Lorentz frames.

We now insert the vector $\vec{A}$ into the second slot of $\mathsf{P}$ in 1.

$$\mathsf{P}\left(\_,\vec{A}\right)=\mathsf{g}\left(\_,\vec{A}\right)+\left(\vec{U}\otimes\vec{U}\right)\left(\_,\vec{A}\right) \quad (12)$$

In components, we have

$$\mathsf{g}\left(\_,\vec{A}\right)_{\alpha}=g_{\alpha\beta}A^{\beta} \quad (13)$$
$$=A_{\alpha} \quad (14)$$
$$\left(\vec{U}\otimes\vec{U}\right)\left(\_,\vec{A}\right)_{\alpha}=U_{\alpha}U_{\beta}A^{\beta} \quad (15)$$
$$=U_{\alpha}\left(\vec{A}\cdot\vec{U}\right) \quad (16)$$

so we have

$$\mathsf{P}\left(\_,\vec{A}\right)=\vec{A}+\left(\vec{A}\cdot\vec{U}\right)\vec{U} \quad (17)$$

In the observer's frame we use 2 and find that

$$\mathsf{P}\left(\_,\vec{A}\right)^{0}=A^{0}+\left(\vec{A}\cdot\vec{U}\right)U^{0} \quad (18)$$
$$=-A_{0}+\left(\vec{A}\cdot\vec{U}\right) \quad (19)$$
$$=-A_{0}+A_{0} \quad (20)$$
$$=0 \quad (21)$$

For the spatial components, $U^{k}=0$ in the observer's frame so

$$\mathsf{P}\left(\_,\vec{A}\right)^{k}=A^{k}+\left(\vec{A}\cdot\vec{U}\right)U^{k} \quad (22)$$
$$=A^{k} \quad (23)$$

Thus $\mathsf{P}\left(\_,\vec{A}\right)$ is the projection of $\vec{A}$ onto the observer's 3-space, since the 0 component $\mathsf{P}\left(\_,\vec{A}\right)^{0}$ of the projection is zero.

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If a projection operator is applied more than once, the applications beyond the first should have no further effect. Consider

$$P_{\alpha\mu}P^{\mu}{}_{\beta} = (g_{\alpha\mu} + U_{\alpha}U_{\mu})\left(g^{\mu}{}_{\beta} + U^{\mu}U_{\beta}\right) \quad (24)$$

$$= g_{\alpha\beta} + U_{\alpha}U_{\beta} + U_{\alpha}U_{\beta} + U_{\alpha}U_{\mu}U^{\mu}U_{\beta} \quad (25)$$

where we've used

$$g^{\mu}{}_{\beta} = \delta_{\mu\beta} \quad (26)$$

We now use the fact that $\vec{U}$ is a 4-velocity, so

$$\vec{U} \cdot \vec{U} = U_{\mu}U^{\mu} = -1 \quad (27)$$

Thus, the last 2 terms in 25 cancel, and we have

$$P_{\alpha\mu}P^{\mu}{}_{\beta} = g_{\alpha\beta} + U_{\alpha}U_{\beta} = P_{\alpha\beta} \quad (28)$$

Applying $\mathsf{P}$ a second time has no further effect, and it behaves like a projection operator.

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Ex 2.10(b) We've seen that the 3-velocity $\boldsymbol{v}$ of the particle as measured by the observer can be written as a 4-vector as

$$\vec{v} = \frac{\vec{p} + \left(\vec{p} \cdot \vec{U}\right)\vec{U}}{-\vec{p} \cdot \vec{U}} \quad (29)$$

From 17, we can set $\vec{A} = \vec{p}$ to get

$$\vec{v} = \frac{\mathsf{P}\left(\_, \vec{p}\right)}{-\vec{p} \cdot \vec{U}} \quad (30)$$