DOPPLER SHIFT DERIVED WITHOUT LORENTZ TRANSFORMATIONS

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Reference: Kip S. Thorne & Roger D. Blandford, *Modern Classical Physics*, Princeton University Press (2017). Exercise 2.11.

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Here we revisit the Doppler effect in relativity. We use the quantum formula for the relation between a photon's energy $\mathcal{E}$ and its frequency ν:

$$\mathcal{E} = h\nu \tag{1}$$

where h is Planck's constant. The goal is to derive the Doppler formula without using Lorentz transformations. We begin with the formula relating energy $\mathcal{E}$, 4-momentum $\vec{p}$ and 4-velocity $\vec{U}$:

$$\mathcal{E} = -\vec{p}\cdot\vec{U} \tag{2}$$

Ex. 2.11(a) We consider an observer at rest in a Lorentz frame who observes an atom moving with ordinary velocity $\boldsymbol{v}$ (as measured by the observer). The atom emits a photon with a 4-momentum $\vec{p}$ towards the observer, who detects it. The Doppler formula is the ratio of the frequencies (and hence the energies, due to 1) of the emitted photon as measured by the emitter and receiver.

The energy given by 2 can be calculated for the emitter, where we use the atom's 4-velocity $\vec{U}_e$, and the receiver, where we use the receiver's 4-velocity $\vec{U}_r$, both measured in the observer's frame. In the observer's frame, he is at rest, so we have

$$\vec{U}_r = (1, \mathbf{0}) \tag{3}$$

where the bold $\mathbf{0}$ is a 3-vector with all components equal to zero.

The atom is moving with 3-velocity $\boldsymbol{v}$, so we have

$$\vec{U}_e = (\gamma, \gamma\boldsymbol{v}) \tag{4}$$

where

$$\gamma = \frac{1}{\sqrt{1-v^2}} \tag{5}$$

For a photon

$$p^0 = |\boldsymbol{p}| \tag{6}$$

where $\boldsymbol{p}$ is the 3-momentum. The required ratio for the Doppler effect is then

Remember to use the metric $\eta_{\alpha\beta}$ when calculating $\vec{p}\cdot\vec{U}$.

$$\frac{\nu_r}{\nu_e} = \frac{\mathcal{E}_r}{\mathcal{E}_e} \tag{7}$$

$$= \frac{\vec{p}\cdot\vec{U}_r}{\vec{p}\cdot\vec{U}_e} \tag{8}$$

$$= \frac{-p^0}{-\gamma p^0 + \gamma \boldsymbol{p}\cdot\boldsymbol{v}} \tag{9}$$

$$= \frac{p^0/\gamma}{p^0 - |\boldsymbol{p}|\,\boldsymbol{v}\cdot\boldsymbol{n}} \tag{10}$$

$$= \frac{\sqrt{1-v^2}}{1-\boldsymbol{v}\cdot\boldsymbol{n}} \tag{11}$$

where $\boldsymbol{n}$ is a unit vector in the direction of $\boldsymbol{p}$, that is, the direction of the photon. If $\boldsymbol{v}$ is parallel to $\boldsymbol{n}$, this formula reduces to the usual Doppler effect formula:

$$\frac{\nu_r}{\nu_e} = \sqrt{\frac{1+v}{1-v}} \tag{12}$$

If $\boldsymbol{v}$ is towards the observer, then $\boldsymbol{v}\cdot\boldsymbol{n} = |\boldsymbol{v}| = v > 0$ and $\nu_r > \nu_e$ indicating a blue shift (the frequency increases). If $\boldsymbol{v}$ is away from the observer, $v < 0$ and we get a red shift.

Ex 2.11(b) If the atom emits a particle with rest mass m instead of a photon, and this particle has 4-velocity $\vec{V}$ and ordinary 3-velocity $\boldsymbol{V}$ as measured by the stationary observer, then the 4-momentum of the particle is

$$\vec{p} = m\vec{V} \tag{13}$$

$$= \gamma m (1, \boldsymbol{V}) \tag{14}$$

The 4-velocities of the atom and observer are as in part (a), as given in 3 and 4.

The ratio of received energy to emitted energy is then

$$\frac{\mathcal{E}_r}{\mathcal{E}_e} = \frac{\vec{p}\cdot\vec{U}_r}{\vec{p}\cdot\vec{U}_e} \tag{15}$$

$$= \frac{-\gamma m}{-\gamma^2 m + \gamma^2 m \boldsymbol{V}\cdot\boldsymbol{v}} \tag{16}$$

$$= \frac{\sqrt{1-v^2}}{1-\boldsymbol{v}\cdot\boldsymbol{V}} \tag{17}$$