LORENTZ BOOSTS

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Reference: Kip S. Thorne & Roger D. Blandford, *Modern Classical Physics*, Princeton University Press (2017). Exercise 2.12.

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The Lorentz transformation for a boost along the x axis is given by

$$\left[L^{\alpha}{}_{\overline{\mu}}\right]=\left[\begin{array}{cccc}\gamma & \beta\gamma & 0 & 0\\ \beta\gamma & \gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{array}\right] \tag{1}$$

where

$$\gamma=\frac{1}{\sqrt{1-\beta^{2}}} \tag{2}$$

and β is the relative speed of the two inertial frames, expressed as a fraction of c, the speed of light. The inverse transformation reverses the direction of relative motion, so β becomes $-\beta$ and γ remains the same:

$$\left[L^{\overline{\mu}}{}_{\alpha}\right]=\left[\begin{array}{cccc}\gamma & -\beta\gamma & 0 & 0\\ -\beta\gamma & \gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{array}\right] \tag{3}$$

We can verify that these two matrices are inverses of each other by direction computation.

$$\left[L^{\alpha}{}_{\overline{\mu}}\right]\left[L^{\overline{\mu}}{}_{\alpha}\right]=\begin{bmatrix}\gamma & \beta\gamma & 0 & 0\\ \beta\gamma & \gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\gamma & -\beta\gamma & 0 & 0\\ -\beta\gamma & \gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} \tag{4}$$

$$=\begin{bmatrix}\gamma^2\left(1-\beta^2\right) & 0 & 0 & 0\\ 0 & \gamma^2\left(1-\beta^2\right) & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} \tag{5}$$

$$=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} \tag{6}$$

where we used 2 to get the last line.

We can also verify that a Lorentz transformation of the metric tensor $g_{\alpha\beta}$ leaves it unchanged. The metric tensor transforms as

$$g_{\overline{\mu}\overline{\nu}}=g_{\alpha\beta}L^{\alpha}{}_{\overline{\mu}}L^{\beta}{}_{\overline{\nu}} \tag{7}$$

Expanding and saving only the non-zero terms in the sums, we have

$$g_{\overline{0}\overline{0}}=g_{\alpha\beta}L^{\alpha}{}_{\overline{0}}L^{\beta}{}_{\overline{0}} \tag{8}$$

$$=g_{00}L^{0}{}_{\overline{0}}L^{0}{}_{\overline{0}}+g_{11}L^{1}{}_{\overline{0}}L^{1}{}_{\overline{0}} \tag{9}$$

$$=-\gamma^2+\beta^2\gamma^2 \tag{10}$$

$$=-1 \tag{11}$$

$$g_{\overline{0}\overline{1}}=g_{\alpha\beta}L^{\alpha}{}_{\overline{0}}L^{\beta}{}_{\overline{1}} \tag{12}$$

$$=g_{00}L^{0}{}_{\overline{0}}L^{0}{}_{\overline{1}}+g_{11}L^{1}{}_{\overline{0}}L^{1}{}_{\overline{1}} \tag{13}$$

$$=-\gamma^2\beta+\gamma^2\beta \tag{14}$$

$$=0 \tag{15}$$

$$g_{\overline{1}\overline{1}}=g_{\alpha\beta}L^{\alpha}{}_{\overline{1}}L^{\beta}{}_{\overline{1}} \tag{16}$$

$$=g_{00}L^{0}{}_{\overline{1}}L^{0}{}_{\overline{1}}+g_{11}L^{1}{}_{\overline{1}}L^{1}{}_{\overline{1}} \tag{17}$$

$$=-\beta^2\gamma^2+\gamma^2 \tag{18}$$

$$=+1 \tag{19}$$

with other elements worked out similarly. The final result is that the metric is invariant under a Lorentz transformation (or at least, under a boost in the

x direction, but the conclusion is true in general, although we don't show that here).

We can actually do the transformation 7 as a sequence of two matrix multiplications. A matrix multiplication requires that we sum the product of the column index in the first matrix multiplied into the row index of the second, so we have for the first multiplication:

$$\left[g_{\alpha\beta}\right]\left[L^{\beta}{}_{\overline{\nu}}\right]=\left[\begin{array}{cccc}-\gamma & -\beta\gamma & 0 & 0\\ \beta\gamma & \gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{array}\right] \tag{20}$$

The row index of this product is α and the column index is $\overline{\nu}$. We now multiply this into $L^{\alpha}{}_{\overline{\mu}}$ but to do this we need the α index in the first matrix to be a column index, so we take the transpose. That is

$$\left[g_{\alpha\beta}L^{\alpha}{}_{\overline{\mu}}L^{\beta}{}_{\overline{\nu}}\right]=\left(\left[g_{\alpha\beta}\right]\left[L^{\beta}{}_{\overline{\nu}}\right]\right)^{T}\left[L^{\alpha}{}_{\overline{\mu}}\right] \tag{21}$$

$$=\left[\begin{array}{cccc}-\gamma & \beta\gamma & 0 & 0\\ -\beta\gamma & \gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{array}\right]\left[\begin{array}{cccc}\gamma & \beta\gamma & 0 & 0\\ \beta\gamma & \gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{array}\right] \tag{22}$$

$$=\left[\begin{array}{cccc}-\gamma^{2}\left(1-\beta^{2}\right) & -\gamma^{2}\beta+\gamma^{2}\beta & 0 & 0\\ -\gamma^{2}\beta+\gamma^{2}\beta & \gamma^{2}\left(1-\beta^{2}\right) & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{array}\right] \tag{23}$$

$$=\left[\begin{array}{cccc}-1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{array}\right] \tag{24}$$

$$=\left[g_{\overline{\mu\nu}}\right] \tag{25}$$

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