# GENERAL BOOSTS AND ROTATIONS

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Reference: Kip S. Thorne & Roger D. Blandford, *Modern Classical Physics*, Princeton University Press (2017). Exercise 2.13.

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Ex. 2.13(a) We are given the matrix

$$\Lambda=\left[\begin{array}{cccc}\gamma & -\beta\gamma n^{1} & -\beta\gamma n^{2} & -\beta\gamma n^{3}\\ -\beta\gamma n^{1} & 1+(\gamma-1)\left(n^{1}\right)^{2} & (\gamma-1)\,n^{1}n^{2} & (\gamma-1)\,n^{1}n^{3}\\ -\beta\gamma n^{2} & (\gamma-1)\,n^{1}n^{2} & 1+(\gamma-1)\left(n^{2}\right)^{2} & (\gamma-1)\,n^{2}n^{3}\\ -\beta\gamma n^{3} & (\gamma-1)\,n^{1}n^{3} & (\gamma-1)\,n^{2}n^{3} & 1+(\gamma-1)\left(n^{3}\right)^{2}\end{array}\right] \tag{1}$$

and asked to show that this is a Lorentz boost along the direction $\boldsymbol{n}$ (a unit vector). We've already done this, so see the earlier post.

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Ex. 2.13(b) We now have a matrix

$$\left[L^{\alpha}{}_{\overline{\mu}}\right]=\left[\begin{array}{cccc}1 & 0 & 0 & 0\\ 0 & & & \\ 0 & & [R_{ij}] & \\ 0 & & & \end{array}\right] \tag{2}$$

where $[R_{ij}]$ is a 3-d rotation matrix. Since a rotation matrix preserves the length of any vector $\boldsymbol{v}$, we have

$$|\boldsymbol{v}|^{2}=\boldsymbol{v}^{T}\boldsymbol{v} \tag{3}$$

$$=(R\boldsymbol{v})^{T}(R\boldsymbol{v}) \tag{4}$$

$$=\boldsymbol{v}^{T}R^{T}R\boldsymbol{v} \tag{5}$$

Therefore, we must have

$$R^{T}R=I \tag{6}$$

the identity matrix, and thus

$$R^{T}=R^{-1} \tag{7}$$

so the transpose of a rotation matrix is its inverse.

We can use this fact to show that 2 is a Lorentz transformation. The condition for this to be true is that 2 satisfies T&B's eqn 2.35b, that is

$$g_{\overline{\mu}\overline{\nu}}=g_{\alpha\beta}L^{\alpha}{}_{\overline{\mu}}L^{\beta}{}_{\overline{\nu}} \tag{8}$$

where $g_{\alpha\beta}$ is the metric tensor. We can use an argument similar to the one in the earlier post to show this.

First, since the 0 row and 0 column in 2 is as in the identity matrix (that is, a 1 in the diagonal position and zeroes elsewhere), it leaves elements of $g$ with a 0 index in either position unchanged. In the 3 spatial coordinates, the components of $g$ are just the 3-d identity matrix. Thus we have

$$\left[g_{\alpha\beta}L^{\beta}{}_{\overline{\nu}}\right]=\left[\begin{array}{cccc}-1 & 0 & 0 & 0\\ 0\\ 0 & & [R_{ij}]\\ 0\end{array}\right] \tag{9}$$

To get the full transformation 8, we take the transpose of 9 and multiply it into 2. We get

$$\begin{aligned}\left[g_{\alpha\beta}L^{\alpha}{}_{\overline{\mu}}L^{\beta}{}_{\overline{\nu}}\right] &= \left[\begin{array}{cccc}-1 & 0 & 0 & 0\\ 0\\ 0 & & [R_{ij}]\\ 0\end{array}\right]^{T}\left[\begin{array}{cccc}1 & 0 & 0 & 0\\ 0\\ 0 & & [R_{ij}]\\ 0\end{array}\right] && (10)\\ &= \left[\begin{array}{cccc}-1 & 0 & 0 & 0\\ 0\\ 0 & & \left[R_{ij}^{T}\right]\\ 0\end{array}\right]\left[\begin{array}{cccc}1 & 0 & 0 & 0\\ 0\\ 0 & & [R_{ij}]\\ 0\end{array}\right] && (11)\\ &= \left[\begin{array}{cccc}-1 & 0 & 0 & 0\\ 0\\ 0 & & \left[R^{T}R\right]\\ 0\end{array}\right] && (12)\\ &= \left[\begin{array}{cccc}-1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{array}\right] && (13)\\ &= \left[g_{\overline{\mu}\overline{\nu}}\right] && (14)\end{aligned}$$

Thus the metric tensor is unchanged by a rotation.

We can also show that the transformation 2 leaves the velocity of the inertial frame unchanged. The velocity of a point in the transformed frame is given by

$$v_{\bar{i}} = \frac{dx_{\bar{i}}}{dx_{\bar{0}}} \tag{15}$$

We have

$$dx_{\bar{0}} = L^{\alpha}{}_{\bar{0}} dx_{\alpha} = dx_0 \tag{16}$$

so that the time interval is preserved between the two frames. For the spatial interval, we have

$$dx_{\bar{i}} = L^{\alpha}{}_{\bar{i}} dx_{\alpha} = R_{ij} dx_j \tag{17}$$

where there is a sum over $j$ in the last term.

The squared magnitude of the velocity thus transforms according to (we use 16, and all repeated indices are summed over 1, 2, 3):

$$\left(\frac{dx_{\bar{i}}}{dx_{\bar{0}}}\right)^2 = R_{ij}\frac{dx_j}{dx_0}R_{ik}\frac{dx_k}{dx_0} \tag{18}$$

$$= R_{ji}^T R_{ik}\frac{dx_j}{dx_0}\frac{dx_k}{dx_0} \tag{19}$$

$$= \delta_{jk}\frac{dx_j}{dx_0}\frac{dx_k}{dx_0} \tag{20}$$

$$= \left(\frac{dx_j}{dx_0}\right)^2 \tag{21}$$

where we used 6 to get the third line. Thus the relative velocity is unchanged by a rotation (which makes sense).