SPACETIME DIAGRAMS

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Reference: Kip S. Thorne & Roger D. Blandford, *Modern Classical Physics*, Princeton University Press (2017). Exercise 2.14.

Post date: 21 Oct 2020.

We now give a few examples to show how spacetime diagrams can illustrate some of the basic facts of special relativity.

First, we need to establish the axes for two different inertial frames, where as usual, the two frames have their x and $\overline{x}$ axes parallel, and the origins coincide, so that $t=0$ is also $\overline{t}=0$ and $x=0$ is also $\overline{x}=0$. From the Lorentz transformations, we have

$$\begin{aligned}\overline{t} &= \gamma(t-\beta x)\\ \overline{x} &= \gamma(x-\beta t)\end{aligned} \tag{1}$$

If we draw the diagram so that the t and x axes are perpendicular, then the $\overline{t}$ axis (corresponding to $\overline{x}=0$) has the equation

$$t=\frac{1}{\beta}x \tag{2}$$

and is therefore a straight line with slope $\frac{1}{\beta}$. Likewise, the $\overline{x}$ axis is found from $\overline{t}=0$ and is

$$t=\beta x \tag{3}$$

and is therefore a straight line with slope β. The angle between the $\overline{t}$ and t axes (and also between the $\overline{x}$ and x axes) is therefore $\theta=\tan^{-1}\beta$. This is shown in Fig. 1.

Ex. 2.14(a) Now suppose we have two events that are simultaneous in the barred frame, as in Fig. 2. Events that are simultaneous in the barred frame lie along a line that is parallel to the $\overline{x}$ axis, such as the red line in Fig. 2. The times of these events as measured in the lab frame can be found by projecting the points onto the t (unbarred) axis as shown by the blue dashed lines. We see that these two events are *not* simultaneous in the lab frame, and that the event that has a lower $\overline{x}$ coordinate has the earlier time t_1 in the lab frame.

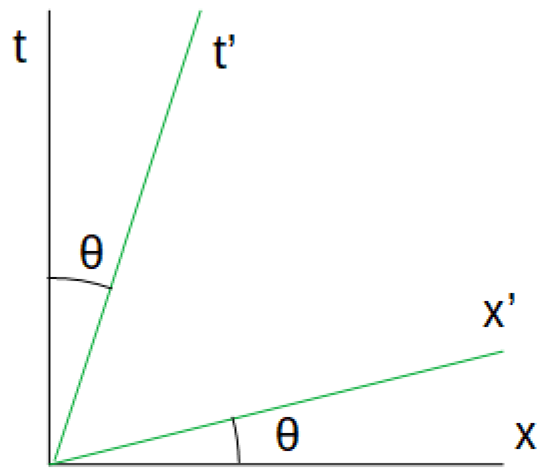

FIGURE 1. Two inertial frames. The barred frame is indicated with primes, since I couldn't get the drawing package to put a bar above a letter.

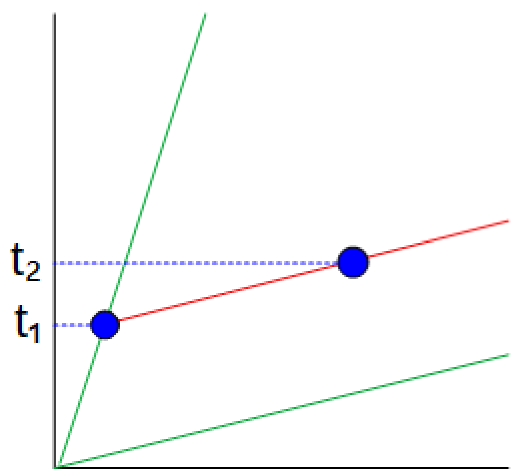

FIGURE 2. Two events simultaneous in the barred frame.

Ex. 2.14(b). Now consider two events that occur at the same place in the barred frame. Such events lie along a line that is parallel to the $\bar{t}$ axis, as shown by the red line in Fig. 3. The locations of these events in the lab frame can be found by projecting the points onto the x axis, and we see that

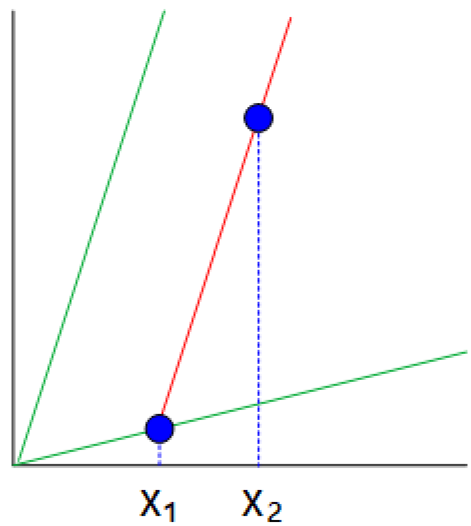

FIGURE 3. Two events at the same location in the barred frame.

the event with the earlier time in the barred frame has the lower location x_1 in the lab frame.

Ex. 2.14(c). To understand the concepts of timelike and spacelike separations, we need the idea of the light cone (Fig. 4). The yellow line shows the path of a photon in a spacetime diagram. This line makes an angle of $\frac{\pi}{4}$ (45°) with both the t and x axes. Since no inertial frame can travel faster than light, the $\bar{t}$ axis of any inertial frame must make an angle of less than $\frac{\pi}{4}$ with the t axis, and the $\bar{x}$ axis must make an angle of less than $\frac{\pi}{4}$ with the x axis.

Thus any two events connected by a line that is parallel to an allowable $\bar{t}$ axis, such as the pair of orange events in Fig. 5, are events that can be traversed by an object travelling more slowly than light, which means they must always have a non-zero time interval between them. In terms of the invariant interval, the separation of these two events must satisfy $\Delta s^2 = -\Delta t^2 + \Delta x^2 < 0$. Since it possible to find a frame in which these two events lie along a line that is parallel to that frame's $\bar{t}$ axis, in that frame, the two events occur at the same spatial position, so that $\Delta\bar{x} = 0$.

Ex. 2.14(d)

Similarly, if two events, such as the pair of purple events in Fig. 5, lie along a line that is parallel to an axis making an angle of less than $\frac{\pi}{4}$ with the x axis, then it is not possible to traverse these events moving more slowly

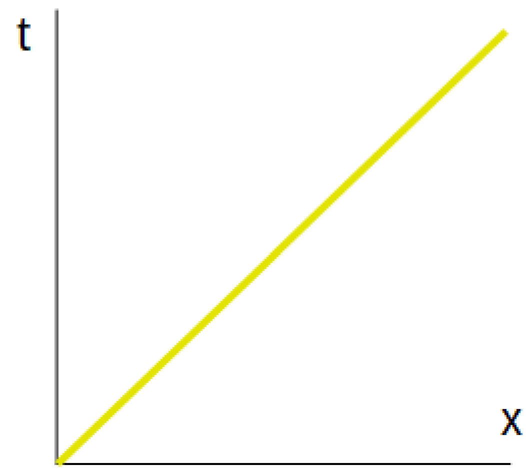

FIGURE 4. The light cone.

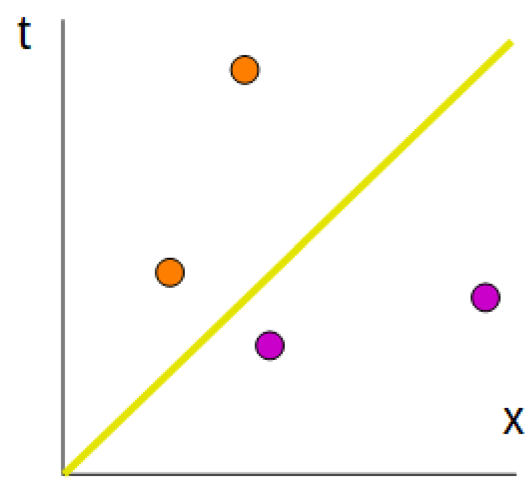

FIGURE 5. Timelike (orange) and spacelike (purple) pairs of events.

than light, and thus these events cannot influence each other. The events are spacelike, and the interval between them satisfies $\Delta s^2 = -\Delta t^2 + \Delta x^2 > 0$. It is thus possible to find an inertial frame where $\Delta t = 0$, or in other words, a frame where the events are connected by a line that is parallel to that frame's $\overline{x}$ axis.

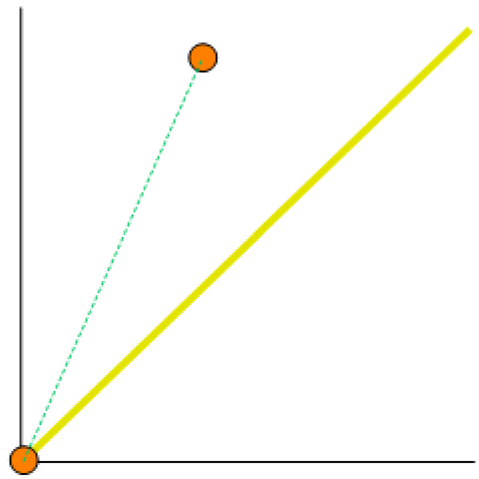

FIGURE 6. Two ticks of a clock.

Ex. 2.14(e). Time dilation. Now consider a pair of events such as the orange events in Fig. 6 that represent successive ticks of a clock at rest in a moving frame.

Since the clock is at rest in that frame, the two events lie along that frame's $\bar{t}$ axis (green dashed line). The separation of these two events is

$$\Delta s^2 = -\Delta \bar{t}^2 \tag{4}$$

In the lab frame, the invariant interval must be the same, so we have

$$\Delta s^2 = -\Delta t^2 + \Delta x^2 \tag{5}$$

From 2 we have, for the $\bar{t}$ axis

$$\Delta x = \beta \Delta t \tag{6}$$

so we get

$$-\Delta \bar{t}^2 = -\Delta t^2 \left(1 - \beta^2\right) \tag{7}$$

or

$$\Delta \bar{t} = \sqrt{1 - \beta^2} \Delta t \tag{8}$$

That is, the moving clock appears to be ticking more slowly when viewed from the lab frame.

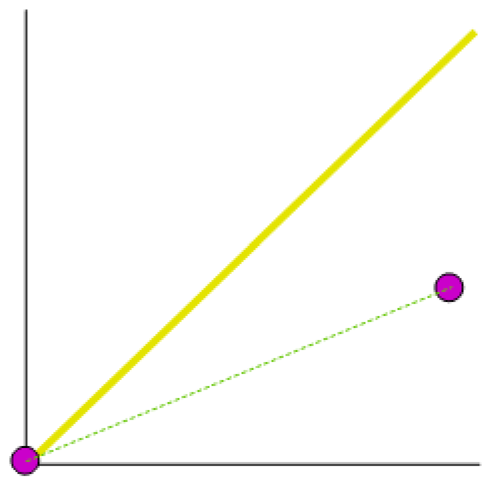

FIGURE 7. An object stationary in the moving frame.

Ex. 2.14(f) . Length contraction. Now suppose we have an object with a fixed size in the moving frame. We can represent this object by two 'events', with one event corresponding to the location of one end of the object at a given time. In the moving frame, this pair of events will always have the same time, so we can represent it by the pair of purple events in Fig. 7. In this frame, the separation of the two ends of the object is given by the invariant interval

$$\Delta s^2 = \Delta \overline{x}^2 \tag{9}$$

In the lab frame, we have

$$\Delta s^2 = -\Delta t^2 + \Delta x^2 \tag{10}$$

and from 3, we have for the $\overline{x}$ axis that

$$\Delta t = \beta \Delta x \tag{11}$$

so

$$\Delta \overline{x}^2 = \left(-\beta^2 + 1\right) \Delta x^2 \tag{12}$$

or

$$\Delta \overline{x} = \sqrt{1 - \beta^2} \Delta x \tag{13}$$

Thus the object appears shortened when viewed from the lab frame.