TWINS PARADOX

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Reference: Kip S. Thorne & Roger D. Blandford, *Modern Classical Physics*, Princeton University Press (2017). Exercise 2.16.

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Ex 2.16(a) The 4-acceleration is defined as

$$\vec{a} \equiv \frac{d\vec{u}}{d\tau} \tag{1}$$

where $\vec{u}$ is the 4-velocity and τ is the proper time. From the relation

$$\vec{u}\cdot\vec{u} = -1 \tag{2}$$

we take the derivative to get

$$\frac{d}{d\tau}(\vec{u}\cdot\vec{u}) = 2\vec{a}\cdot\vec{u} = 0 \tag{3}$$

so that the 4-acceleration is orthogonal (in the 4-dim spacetime sense) to the 4-velocity.

If we look at the object in its own frame, then we can use an instantaneous co-moving frame, in which

$$\vec{u} = (1,0,0,0) \tag{4}$$

we have

$$\vec{a}\cdot\vec{u} = -a^0u^0 + \boldsymbol{a}\cdot\boldsymbol{u} \tag{5}$$

where a bold font indicates a 3-space vector. Since $\boldsymbol{u} = (0,0,0)$ in the object's co-moving frame, we have

$$a^0u^0 = 0 \tag{6}$$

Since $u^0 = 1$, we must have

$$a^0 = 0 \tag{7}$$

and therefore

$$\vec{a}\cdot\vec{a} = -\left(a^0\right)^2 + |\boldsymbol{a}|^2 \tag{8}$$
$$= |\boldsymbol{a}|^2 \tag{9}$$

so the magnitude of the 3-acceleration is the same as the magnitude of the 4-acceleration:

$$|\boldsymbol{a}| = \sqrt{\vec{a}\cdot\vec{a}} \tag{10}$$

Ex. 2.16(b) We now consider two observers, Methuselah and Florence. Methuselah stays on Earth at rest, while Florence boards a rocket and accelerates away from Earth at a constant rate of g for a time $\frac{T_F}{4}$. We want to find how much time Methuselah thinks has elapsed in the same interval. In Methuselah's frame, the 4-velocity of the rocket is

$$\vec{u}_M = (\gamma, \gamma\boldsymbol{v}) \tag{11}$$

where

$$\gamma = \frac{1}{\sqrt{1-\boldsymbol{v}^2}} \tag{12}$$

Since the rocket is accelerating, $\boldsymbol{v}$ is no longer a constant. We'll restrict motion to one dimension, so

$$\boldsymbol{v} = (v, 0, 0) \tag{13}$$

If we take the derivative, we get

$$\frac{d\gamma}{d\tau} = \frac{v}{(1-v^2)^{3/2}}\frac{dv}{d\tau} = \gamma^3 v\frac{dv}{d\tau} \tag{14}$$

$$\frac{d\vec{u}_M}{d\tau} = \left(\gamma^3 v, \gamma^3 v^2 + \gamma, 0, 0\right)\frac{dv}{d\tau} \tag{15}$$

We can simplify this with the relation

$$\gamma^3 v^2 + \gamma = \gamma^3 v^2 + \frac{\gamma^3}{\gamma^2} \tag{16}$$
$$= \gamma^3\left(v^2 + 1 - v^2\right) \tag{17}$$
$$= \gamma^3 \tag{18}$$

so we have

$$\frac{d\vec{u}_M}{d\tau} = \left(\gamma^3 v, \gamma^3, 0, 0\right) \frac{dv}{d\tau} \tag{19}$$

where $\frac{dv}{d\tau}$ is the acceleration as measured by Methuselah.

From 10, we know that the magnitude of the 3-acceleration in Florence's frame is the magnitude of the 4-acceleration, so we have

$$|\boldsymbol{a}_F| = \sqrt{-\gamma^6 v^2 + \gamma^6} \frac{dv}{d\tau} \tag{20}$$

$$= \gamma^2 \frac{dv}{d\tau} \tag{21}$$

The magnitude of Florence's acceleration is constant at one Earth gravity g, so we have

$$\frac{dv}{d\tau} = \left(1 - v^2\right) g \tag{22}$$

If Florence accelerates at a constant rate (according to her), then as seen by Methuselah, her speed will asymptotically approach the speed of light. That is, Methuselah will see Florence's acceleration getting smaller, since the rate at which her speed is changing must be getting asymptotically close to zero. So as $v \to 1$, $\frac{dv}{d\tau} \to 0$.

We can integrate this equation, starting with

$$\int \frac{dv}{1 - v^2} = \int g \, d\tau \tag{23}$$

which gives us

$$\tanh^{-1} v = gt|_0^\tau \tag{24}$$

$$= g\tau \tag{25}$$

From this, we have

$$\gamma = \frac{1}{\sqrt{1 - \tanh^2(g\tau)}} \tag{26}$$

$$= \frac{1}{\sqrt{\text{sech}^2(g\tau)}} \tag{27}$$

$$= \cosh(g\tau) \tag{28}$$

The relation between the rocket's proper time τ and the Earth time t is

$$\frac{dt}{d\tau} = \gamma \tag{29}$$

so we can integrate 28 to get

$$t = \int_0^{\frac{T_F}{4}} \cosh(g\tau)\, d\tau \tag{30}$$

$$= \frac{1}{g}\sinh\left(\frac{gT_F}{4}\right) \tag{31}$$

In the problem, Florence's journey consists of an outward leg of time $\frac{T_F}{4}$, then a backwards leg of time $\frac{T_F}{2}$ and then a final forward leg of time $\frac{T_F}{4}$, all with the same acceleration. Thus the total time is 4 times the above result, and we have

$$T_M = \frac{4}{g}\sinh\left(\frac{gT_F}{4}\right) \tag{32}$$

Ex. 2.16(c) If we take $g = 9.8\text{ m}\cdot\text{s}^{-2}$ and convert it to geometric units, we divide by $c = 3\times 10^8\text{ m s}^{-1}$ to get

$$g = \frac{9.8}{3\times 10^8}\text{ s}^{-1} \tag{33}$$

$$= 3.267\times 10^{-8}\text{ s}^{-1} \tag{34}$$

$$= \left(3.267\times 10^{-8}\right)\left(3600\text{ s hr}^{-1}\right)\left(24\text{ hr day}^{-1}\right)\left(365.25\text{ days yr}^{-1}\right) \tag{35}$$

$$= 1.031\text{ yr}^{-1} \tag{36}$$

Using this value in 32, we can draw a plot of the function

$$T_M = 3.88\sinh(0.2578T_F) \tag{37}$$

From Fig. 1, we see that for T_M to be around the age of the universe at 14 billions years, Florence's time is only about 75 years. Of course, the amount of fuel that would be required to keep a rocket accelerating for 75 years is enormous.

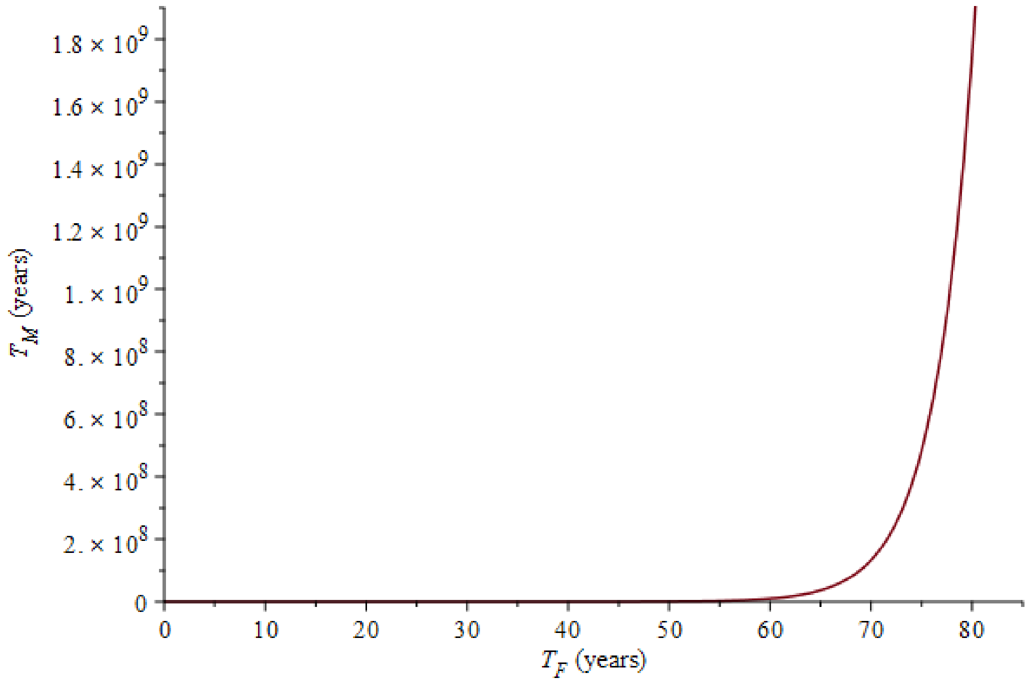

FIGURE 1. Methuselah's time as a function of Florence's time.