# SPACETIME VOLUMES

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Reference: Kip S. Thorne & Roger D. Blandford, *Modern Classical Physics*, Princeton University Press (2017). Section 2.12.1.

Post date: 29 Oct 2020.

In Section 2.12.1, T&B introduce the concept of a 4-dimensional spacetime volume using the 4-dimensional version of the Levi-Civita tensor. The discussion requires a bit of clarification (at least for me). First, we need to be sure of the definition of $\epsilon_{\alpha\beta\gamma\delta}$, which is given in T&B's eqn 2.43:

$$\epsilon_{\alpha\beta\gamma\delta}=+1 \tag{1}$$

if $\alpha,\beta,\gamma,\delta$ is an even permutation of 0, 1, 2, 3. Swapping any two indices reverses the sign, and we get zero if any two indices are equal. We also need to recall the definition of the components of a 4-vector, as given by eqns 2.23a and 2.23b. The contravariant components $A^{\alpha}$ of a 4-vector $\vec{A}$ are given by

$$\vec{A}=A^{\alpha}\vec{e}_{\alpha} \tag{2}$$

where $\vec{e}_{\alpha}$ are the orthonormal basis vectors in spacetime. The contravariant components are given by scalar products, so we have

$$A_{\alpha}\equiv\vec{A}\left(\vec{e}_{\alpha}\right)=\vec{A}\cdot\vec{e}_{\alpha} \tag{3}$$

In particular, due to the Minkowski metric $\eta_{\alpha\beta}$, we have

$$\begin{aligned}\vec{e}_0\cdot\vec{e}_0&=-1\\ \vec{e}_i\cdot\vec{e}_i&=+1\end{aligned} \tag{4}$$

Therefore, for the time component of a 4-vector, we have

$$A_0=\vec{A}\cdot\vec{e}_0=A^{\alpha}\vec{e}_{\alpha}\cdot\vec{e}_0=A^0\vec{e}_0\cdot\vec{e}_0=-A^0 \tag{5}$$

We can now return to the definition of a spacetime volume of a 4-dim parallelipiped with legs $\vec{A}$, $\vec{B}$, $\vec{C}$ and $\vec{D}$, which is given by eqn 2.52 as

$$\text{4-volume}=\epsilon_{\alpha\beta\gamma\delta}A^{\alpha}B^{\beta}C^{\gamma}D^{\delta}=\boldsymbol{\epsilon}\left(\vec{A},\vec{B},\vec{C},\vec{D}\right) \tag{6}$$

For an infinitesimal 4-volume with sides along the four basis vectors, we have

$$\begin{aligned}\vec{A} &= dt\vec{e}_0 \\ \vec{B} &= dx\vec{e}_1 \\ \vec{C} &= dy\vec{e}_2 \\ \vec{D} &= dz\vec{e}_3\end{aligned} \tag{7}$$

The volume element $d\Sigma$ (a scalar) is then given by 6:

$$d\Sigma = dt\ dx\ dy\ dz\ \epsilon_{\alpha\beta\gamma\delta}\vec{e}_0^{\alpha}\vec{e}_1^{\beta}\vec{e}_2^{\gamma}\vec{e}_3^{\delta} \tag{8}$$

Since the basis vectors are orthonormal, we have

$$\begin{aligned}\vec{e}_0 &= (1,0,0,0) \\ \vec{e}_1 &= (0,1,0,0) \\ \vec{e}_2 &= (0,0,1,0) \\ \vec{e}_3 &= (0,0,0,1)\end{aligned} \tag{9}$$

so the volume element is

$$d\Sigma = dt\ dx\ dy\ dz\ \epsilon_{0123} = dt\ dx\ dy\ dz \tag{10}$$

Now things get a bit tricky. We can define a 3-dimensional parallelepiped within 4-dim spacetime in analogy with the definition of a parallelogram in 3-dim space. In 3-dim space, the area of a parallelogram is assigned a direction. If $\boldsymbol{A}$ and $\boldsymbol{B}$ (3-dim vectors) define the legs of the parallelogram, the area is given by the cross product

$$\text{area} = \boldsymbol{A} \times \boldsymbol{B} \tag{11}$$

Note that the area has both a magnitude ($|\boldsymbol{A} \times \boldsymbol{B}| = AB\sin\theta$) *and* a direction, given by the right hand rule. If we swap the order of the sides in the area definition, we get the same magnitude but the opposite direction. A cross product can also be written using the 3-dim Levi-Civita tensor as

$$(\boldsymbol{A} \times \boldsymbol{B})_k = \epsilon_{ijk}A^iB^j \tag{12}$$

In spacetime, we define a 3-dim parallelepiped in analogy with this by using the 4-dim Levi-Civita tensor 1. Just as the area of a parellelogram in 3-dim space is a 3-dim vector, the volume of a parallelepiped in spacetime is a spacetime (4-dim) vector $\vec{\Sigma}$, defined as

$$\vec{\Sigma}(\_) \equiv \boldsymbol{\epsilon}\left(\_,\vec{A},\vec{B},\vec{C}\right) \tag{13}$$

$$\Sigma_\mu = \epsilon_{\mu\beta\gamma\delta}A^\alpha B^\beta C^\gamma \tag{14}$$

We now consider T&B's Fig. 2.10a, in which a 3-dim volume with legs $\Delta x\vec{e}_x$, $\Delta y\vec{e}_y$, $\Delta z\vec{e}_z$. Its volume is the vector

$$\vec{\Sigma} = \boldsymbol{\epsilon}\left(\_,\Delta x\vec{e}_x,\Delta y\vec{e}_y,\Delta z\vec{e}_z\right) \tag{15}$$

$$= \Delta x\Delta y\Delta z\boldsymbol{\epsilon}\left(\_,\vec{e}_x,\vec{e}_y,\vec{e}_z\right) \tag{16}$$

$$\equiv \Delta V\boldsymbol{\epsilon}\left(\_,\vec{e}_x,\vec{e}_y,\vec{e}_z\right) \tag{17}$$

where the 3-dim volume is

$$\Delta V \equiv \Delta x\Delta y\Delta z \tag{18}$$

T&B then say that "it is easy to see" (one of most hated phrases in physics textbooks) that

$$\vec{\Sigma} = -\Delta V\vec{e}_0 \tag{19}$$

At first glance, the obvious question is "why the minus sign"? To see this, we look back at the definitions 2 and 3 for the components of a 4-vector. If we compare 17 with 14 we see that

$$\Sigma_0 = \epsilon_{0123}\Delta x\Delta y\Delta z = +\Delta V \tag{20}$$

However, to get this from the vector representation of $\vec{\Sigma}$, we need to look at 3, where we have

$$\Sigma_0 = \vec{\Sigma}\cdot\vec{e}_0 \tag{21}$$

If we plug 19 into this and use 4, we have

$$\Sigma_0 = -\Delta V\vec{e}_0\cdot\vec{e}_0 = +\Delta V \tag{22}$$

so the equation 19 is, in fact, correct. Note that, because raising or lowering a 0 index changes the sign, we also have

$$\Sigma^0 = -\Delta V \tag{23}$$

so from 2 this also gives us

$$\vec{\Sigma} = -\Delta V\vec{e}_0 \tag{24}$$

The seemingly mysterious minus sign is a consequence of the Minkowski metric.

The same argument applies to T&B's other example, given in Fig. 2.10b. Here, they use a 3-dim volume with legs $\Delta t\vec{e}_0$, $\Delta y\vec{e}_y$, $\Delta z\vec{e}_z$. The volume 14 is then

$$\vec{\Sigma}=\boldsymbol{\epsilon}\left(\_,\Delta t\vec{e}_0,\Delta y\vec{e}_y,\Delta z\vec{e}_z\right) \tag{25}$$

In this case, the only non-zero component of $\vec{\Sigma}$ is in the $x$ direction, so we have

$$\Sigma_1=\epsilon_{1023}\Delta t\Delta y\Delta z=-\epsilon_{0123}\Delta t\Delta y\Delta z\equiv-\Delta t\Delta A \tag{26}$$

with $\Delta A\equiv\Delta y\Delta z$. This can be written as

$$\vec{\Sigma}=-\Delta t\Delta A\vec{e}_x \tag{27}$$

In this case, the covariant component is, from 3 and 4

$$\Sigma_1=-\Delta t\Delta A\vec{e}_x\cdot\vec{e}_x=-\Delta t\Delta A \tag{28}$$

In this case, the negative sign comes from swapping the positions of the $t$ and $x$ directions in $\boldsymbol{\epsilon}$, rather than from the Minkowski metric.

## PINGBACKS

Pingback: Fluid in an inertial frame