

DELTA FUNCTION: A COUPLE OF ALTERNATIVE DERIVATIONS

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References: Anthony Zee, *Quantum Field Theory in a Nutshell*, 2nd edition (Princeton University Press, 2010) - Chapter I.2, Appendix 1.

We've been using the Dirac delta function frequently on this blog, and in some cases we've been using what looks like a dodgy formula in the form of an integral over a complex exponential:

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ikx} dk \quad (1)$$

In fact, this formula can be made to look more credible by the following argument. Suppose we start with a finite integral:

$$d_K(x) \equiv \frac{1}{2\pi} \int_{-\frac{K}{2}}^{\frac{K}{2}} e^{ikx} dk \quad (2)$$

$$= \frac{1}{2\pi ix} e^{ikx} \Big|_{-\frac{K}{2}}^{\frac{K}{2}} \quad (3)$$

$$= \frac{1}{\pi x} \sin\left(\frac{Kx}{2}\right) \quad (4)$$

As $\lim_{x \rightarrow 0} \frac{\sin(ax)}{ax} = 1$, this result has a peak of value $K/2\pi$ at $x = 0$ and oscillates on either side of the origin with decreasing amplitude as you get farther from the origin. Also, $d_K(x) = 0$ for the first time at $x = \pm \frac{2\pi}{K}$ on either side of the origin.

Using the standard integral

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi \quad (5)$$

we find that

$$\int_{-\infty}^{\infty} d_K(x) dx = 1 \quad (6)$$

Note that this result is independent of K , and remains true as $K \rightarrow \infty$. In this limit, the spike at $x = 0$ becomes infinitely large, and the width of the

spike becomes infinitesimal. Thus we can define the delta function as this limit:

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dk \quad (7)$$

Another representation of $\delta(x)$ is the formula

$$\delta(x) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\pi} \frac{\varepsilon}{x^2 + \varepsilon^2} \quad (8)$$

For any nonzero value of ε , this function has a peak of height $\frac{1}{\pi\varepsilon}$ at $x = 0$, and its integral is

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\varepsilon dx}{x^2 + \varepsilon^2} = \frac{1}{\pi} \arctan\left(\frac{x}{\varepsilon}\right) \Big|_{-\infty}^{\infty} \quad (9)$$

$$= 1 \quad (10)$$

As $\varepsilon \rightarrow 0$, the peak height at $x = 0$ becomes infinite, and the peak width becomes zero, since for any $x \neq 0$ $\delta(x) \rightarrow 0$ as $\varepsilon \rightarrow 0$, so it satisfies the requirements of a delta function.

PINGBACKS

Pingback: Delta function - Fourier transform