DELTA FUNCTION: A COUPLE OF ALTERNATIVE DERIVATIONS

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References: Anthony Zee, *Quantum Field Theory in a Nutshell*, 2nd edition (Princeton University Press, 2010) - Chapter I.2, Appendix 1.

We've been using the Dirac delta function frequently on this blog, and in some cases we've been using what looks like a dodgy formula in the form of an integral over a complex exponential:

$$\delta(x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{ikx}dk \tag{1}$$

In fact, this formula can be made to look more credible by the following argument. Suppose we start with a finite integral:

$$\begin{aligned}
d_K(x) &\equiv \frac{1}{2\pi}\int_{-\frac{K}{2}}^{\frac{K}{2}}e^{ikx}dk &(2)\\
&= \frac{1}{2\pi ix}\left.e^{ikx}\right|_{-\frac{K}{2}}^{\frac{K}{2}} &(3)\\
&= \frac{1}{\pi x}\sin\left(\frac{Kx}{2}\right) &(4)
\end{aligned}$$

As $\lim_{x\to0}\frac{\sin(ax)}{ax}=1$, this result has a peak of value $K/2\pi$ at $x=0$ and oscillates on either side of the origin with decreasing amplitude as you get farther from the origin. Also, $d_K(x)=0$ for the first time at $x=\pm\frac{2\pi}{K}$ on either side of the origin.

Using the standard integral

$$\int_{-\infty}^{\infty}\frac{\sin x}{x}dx=\pi \tag{5}$$

we find that

$$\int_{-\infty}^{\infty}d_K(x)\,dx=1 \tag{6}$$

Note that this result is independent of K, and remains true as $K\to\infty$. In this limit, the spike at $x=0$ becomes infinitely large, and the width of the spike becomes infinitesimal. Thus we can define the delta function as this limit:

$$\delta\left(x\right)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ikx}dk \tag{7}$$

Another representation of $\delta\left(x\right)$ is the formula

$$\delta\left(x\right)=\lim_{\varepsilon\to0}\frac{1}{\pi}\frac{\varepsilon}{x^{2}+\varepsilon^{2}} \tag{8}$$

For any nonzero value of ε, this function has a peak of height $\frac{1}{\pi\varepsilon}$ at $x=0$, and its integral is

$$\begin{aligned}
\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\varepsilon\,dx}{x^{2}+\varepsilon^{2}} &= \frac{1}{\pi}\arctan\left(\frac{x}{\varepsilon}\right)\Big|_{-\infty}^{\infty} && (9)\\
&= 1 && (10)
\end{aligned}$$

As $\varepsilon\to0$, the peak height at $x=0$ becomes infinite, and the peak width becomes zero, since for any $x\neq0$ $\delta\left(x\right)\to0$ as $\varepsilon\to0$, so it satisfies the requirements of a delta function.

PINGBACKS

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