

GAUSSIAN INTEGRALS: AVERAGES OF POWERS OF X

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References: Anthony Zee, *Quantum Field Theory in a Nutshell*, 2nd edition (Princeton University Press, 2010) - Chapter I.2, Appendix 2.

The Gaussian integral can be used to define averages of powers of the integration variable x . The average is defined as

$$\langle x^{2n} \rangle \equiv \frac{\int_{-\infty}^{\infty} dx e^{-ax^2/2} x^{2n}}{\int_{-\infty}^{\infty} dx e^{-ax^2/2}} \quad (1)$$

$$= \sqrt{\frac{a}{2\pi}} \int_{-\infty}^{\infty} dx e^{-ax^2/2} x^{2n} \quad (2)$$

We look only at even powers of x since the average of all odd powers is zero, as the integrand is odd. Rather than working out all the integrals, we can actually find $\langle x^{2n} \rangle$ by differentiating the original Gaussian integral with respect to the parameter a . We have

$$G = \int_{-\infty}^{\infty} dx e^{-ax^2/2} \quad (3)$$

$$= \sqrt{\frac{2\pi}{a}} \quad (4)$$

$$-2 \frac{dG}{da} = \int_{-\infty}^{\infty} dx e^{-ax^2/2} x^2 \quad (5)$$

$$= \sqrt{2\pi} \frac{1}{a^{3/2}} \quad (6)$$

$$(-2)^2 \frac{d^2 G}{da^2} = \int_{-\infty}^{\infty} dx e^{-ax^2/2} x^4 \quad (7)$$

$$= \sqrt{2\pi} \frac{3}{a^{5/2}} \quad (8)$$

$$(-2)^n \frac{d^n G}{da^n} = \sqrt{2\pi} \frac{(2n-1)!!}{a^{(2n+1)/2}} \quad (9)$$

where the double factorial is defined as

$$(2n-1)!! \equiv (2n-1)(2n-3)\dots 3 \times 1 \quad (10)$$

Putting this result into 2 we get

$$\langle x^{2n} \rangle = \frac{(2n-1)!!}{a^n} \quad (11)$$

We can also get this result from the variant Gaussian integral, obtained by completing the square in the exponent:

$$\int_{-\infty}^{\infty} dx e^{-ax^2/2+Jx} = \sqrt{\frac{2\pi}{a}} e^{J^2/2a} \quad (12)$$

If we take the derivative of the LHS with respect to J we get

$$\frac{d^{2n}}{dJ^{2n}} \int_{-\infty}^{\infty} dx e^{-ax^2/2+Jx} = \int_{-\infty}^{\infty} dx e^{-ax^2/2+Jx} x^{2n} \quad (13)$$

Setting $J = 0$ and comparing with 2 gives us

$$\langle x^{2n} \rangle = \left. \frac{d^{2n}}{dJ^{2n}} e^{J^2/2a} \right|_{J=0} \quad (14)$$

Although I could leave things here, I decided that it would be interesting to prove that the RHS actually does give 11. This proved to be a bit trickier than I expected, but the derivation is interesting so I'll give it here.

We need to find a general expression for $\frac{d^{2n}}{dJ^{2n}} e^{J^2/2a}$ before setting $J = 0$. To help with this, we can write out the derivative explicitly for the first few values of n . It's actually easier to do the work for $a = 1$; using dimensional analysis it's easy enough to put a back in at the end. We find

n	$\frac{d^{2n}}{dJ^{2n}} e^{J^2/2}$
1	$e^{J^2/2} [1 + J^2]$
2	$e^{J^2/2} [3 + 6J^2 + J^4]$
3	$e^{J^2/2} [15 + 45J^2 + 15J^4 + J^6]$
4	$e^{J^2/2} [105 + 420J^2 + 210J^4 + 28J^6 + J^8]$
5	$e^{J^2/2} [945 + 4725J^2 + 3150J^4 + 630J^6 + 45J^8 + J^{10}]$

We see that the constant term in each case is indeed equal to $(2n-1)!!$. The coefficient of the second highest power of J is $(2n-1)n$. Working backwards in each line, we find that the coefficient of the third highest power is $\frac{1}{2}(2n-1)(2n-3)n(n-1)$, of the fourth highest power is $\frac{1}{3 \times 2}(2n-1)(2n-3)(2n-5)n(n-1)$ and so on. In general, the coefficient of J^{2n-2m} is

$$\frac{n!}{m!(n-m)!} \frac{(2n-1)!!}{(2n-2m-1)!!} = \binom{n}{m} \frac{(2n-1)!!}{(2n-2m-1)!!} \quad (15)$$

Therefore we propose that

$$\frac{d^{2n}}{dJ^{2n}} e^{J^2/2} = e^{J^2/2} \sum_{m=0}^n \binom{n}{m} \frac{(2n-1)!!}{(2n-2m-1)!!} J^{2n-2m} \quad (16)$$

We can prove this in general using induction. We've already established the anchor step, since this formula is true for $n = 1 \dots 5$, so we can assume 16 for some value n and then work from there to prove it's true for $n + 1$. That is, we want to show, starting from 16, that the following is true:

$$\frac{d^{2n+2}}{dJ^{2n+2}} e^{J^2/2} = e^{J^2/2} \sum_{m=0}^{n+1} \binom{n+1}{m} \frac{(2n+1)!!}{(2n-2m+1)!!} J^{2n-2m+2} \quad (17)$$

We need to take the derivative of 16 twice. We get

$$\frac{d^{2n+1}}{dJ^{2n+1}} e^{J^2/2} = e^{J^2/2} \sum_{m=0}^n \binom{n}{m} \frac{(2n-1)!!}{(2n-2m-1)!!} [J^{2n-2m+1} + (2n-2m)J^{2n-2m-1}] \quad (18)$$

$$\begin{aligned} \frac{d^{2n+2}}{dJ^{2n+2}} e^{J^2/2} &= e^{J^2/2} \sum_{m=0}^n \binom{n}{m} \frac{(2n-1)!!}{(2n-2m-1)!!} \times \\ &[J^{2n-2m+2} + (4n-4m+1)J^{2n-2m} + (2n-2m)(2n-2m-1)J^{2n-2m-2}] \end{aligned} \quad (19)$$

We'd like to put this in the form 17, so we can shift the summation index from $m \rightarrow m - 1$ in the second term, and from $m \rightarrow m - 2$ in the third term, thus allowing us to factor out $J^{2n-2m+2}$ from all 3 terms. The limits on the sums will also change, so the second term now has limits of $m = 1 \dots n + 1$ and the third term of $m = 2 \dots n + 2$. We get (putting the exponential on the LHS for convenience):

$$\begin{aligned} e^{-J^2/2} \frac{d^{2n+2}}{dJ^{2n+2}} e^{J^2/2} &= \sum_{m=0}^n \binom{n}{m} \frac{(2n-1)!!}{(2n-2m-1)!!} J^{2n-2m+2} + \\ &\sum_{m=1}^{n+1} \binom{n}{m-1} \frac{(2n-1)!!}{(2n-2m+1)!!} (4n-4m+5) J^{2n-2m+2} + \\ &\sum_{m=2}^{n+2} \binom{n}{m-2} \frac{(2n-1)!!}{(2n-2m+3)!!} (2n-2m+4)(2n-2m+3) J^{2n-2m+2} \end{aligned} \quad (20)$$

We can condense the sums by using the fact that the binomial coefficient $\binom{p}{q}$ is zero if $q < 0$ or $q > p$, so we can extend the lower limits on the second and third sums to 0, and extend the upper limit on the first sum to $n + 1$.

Also, in the third sum, the factor $(2n - 2m + 4)$ is zero when $m = n + 2$, so we can reduce the upper limit on the sum to $n + 1$. Therefore all three sums extend from $m = 0 \dots n + 1$ and we have (cancelling the common factor in the third sum as well):

$$\begin{aligned}
e^{-J^2/2} \frac{d^{2n+2}}{dJ^{2n+2}} e^{J^2/2} &= \sum_{m=0}^{n+1} \binom{n}{m} \frac{(2n-1)!!}{(2n-2m-1)!!} J^{2n-2m+2} + \\
&\quad \sum_{m=0}^{n+1} \binom{n}{m-1} \frac{(2n-1)!!}{(2n-2m+1)!!} (4n-4m+5) J^{2n-2m+2} + \\
&\quad \sum_{m=0}^{n+1} \binom{n}{m-2} \frac{(2n-1)!!}{(2n-2m+1)!!} (2n-2m+4) J^{2n-2m+2}
\end{aligned} \tag{21}$$

To convert the binomial coefficients, we have

$$\binom{n}{m-2} = \frac{m-1}{n-m+2} \binom{n}{m-1} \tag{22}$$

This allows us to combine the second and third sums:

$$\begin{aligned}
e^{-J^2/2} \frac{d^{2n+2}}{dJ^{2n+2}} e^{J^2/2} &= \sum_{m=0}^{n+1} \binom{n}{m} \frac{(2n-1)!!}{(2n-2m-1)!!} J^{2n-2m+2} + \\
&\quad \sum_{m=0}^{n+1} \binom{n}{m-1} \frac{(2n-1)!!}{(2n-2m+1)!!} (4n-2m+3) J^{2n-2m+2}
\end{aligned} \tag{23}$$

Also

$$\binom{n}{m-1} = \frac{m}{n-m+1} \binom{n}{m} \tag{24}$$

Using this, and multiplying the first sum by $\frac{2n-2m+1}{2n-2m+1}$ we can combine it with the second sum to get

$$e^{-J^2/2} \frac{d^{2n+2}}{dJ^{2n+2}} e^{J^2/2} = \sum_{m=0}^{n+1} \binom{n}{m} \frac{(2n-1)!!}{(2n-2m+1)!!} \left(2n-2m+1 + \frac{m(4n-2m+3)}{n-m+1} \right) J^{2n-2m+2} \tag{25}$$

Finally, we have

$$\binom{n}{m} = \frac{n-m+1}{n+1} \binom{n+1}{m} \tag{26}$$

Plugging this in and simplifying, we get

$$\left(2n - 2m + 1 + \frac{m(4n - 2m + 3)}{n - m + 1}\right) \binom{n}{m} = \left(2n - 2m + 1 + \frac{m(4n - 2m + 3)}{n - m + 1}\right) \frac{n - m + 1}{n + 1} \binom{n + 1}{m} \quad (27)$$

$$= (2n + 1) \binom{n + 1}{m} \quad (28)$$

$$e^{-J^2/2} \frac{d^{2n+2}}{dJ^{2n+2}} e^{J^2/2} = \sum_{m=0}^{n+1} \binom{n+1}{m} \frac{(2n+1)!!}{(2n-2m+1)!!} J^{2n-2m+2} \quad (29)$$

QED.

To restore the factors of a , we observe that J^2 has the same dimensions as a (since an exponent must be dimensionless), so the derivative $\frac{d^{2n}}{dJ^{2n}}$ has the dimensions of a^{-n} . Therefore, the term involving J^{2n-2m} must be divided by a^{2n-m} . Thus:

$$\frac{d^{2n}}{dJ^{2n}} e^{J^2/2a} = e^{J^2/2a} \sum_{m=0}^n \binom{n}{m} \frac{(2n-1)!!}{(2n-2m-1)!!} \frac{J^{2n-2m}}{a^{2n-m}} \quad (30)$$

When $J = 0$, this reduces to the $m = n$ term, which is (the double factorial $(-1)!! = 1$, at least according to Maple):

$$\left. \frac{d^{2n}}{dJ^{2n}} e^{J^2/2a} \right|_{J=0} = \frac{(2n-1)!!}{a^n} \quad (31)$$

which is the same as 11.

PINGBACKS

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