GAUSSIAN INTEGRALS: AVERAGES OF POWERS OF X

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References: Anthony Zee, *Quantum Field Theory in a Nutshell*, 2nd edition (Princeton University Press, 2010) - Chapter I.2, Appendix 2.

The Gaussian integral can be used to define averages of powers of the integration variable x. The average is defined as

$$\begin{aligned}\left\langle x^{2n}\right\rangle &\equiv \frac{\int_{-\infty}^{\infty}dxe^{-ax^2/2}x^{2n}}{\int_{-\infty}^{\infty}dxe^{-ax^2/2}} &(1)\\ &= \sqrt{\frac{a}{2\pi}}\int_{-\infty}^{\infty}dxe^{-ax^2/2}x^{2n} &(2)\end{aligned}$$

We look only at even powers of x since the average of all odd powers is zero, as the integrand is odd. Rather than working out all the integrals, we can actually find $\left\langle x^{2n}\right\rangle$ by differentiating the original Gaussian integral with respect to the parameter a. We have

$$\begin{aligned}G &= \int_{-\infty}^{\infty}dxe^{-ax^2/2} &(3)\\ &= \sqrt{\frac{2\pi}{a}} &(4)\\ -2\frac{dG}{da} &= \int_{-\infty}^{\infty}dxe^{-ax^2/2}x^2 &(5)\\ &= \sqrt{2\pi}\frac{1}{a^{3/2}} &(6)\\ (-2)^2\frac{d^2G}{da^2} &= \int_{-\infty}^{\infty}dxe^{-ax^2/2}x^4 &(7)\\ &= \sqrt{2\pi}\frac{3}{a^{5/2}} &(8)\\ (-2)^n\frac{d^nG}{da^n} &= \sqrt{2\pi}\frac{(2n-1)!!}{a^{(2n+1)/2}} &(9)\end{aligned}$$

where the double factorial is defined as

$$(2n-1)!! \equiv (2n-1)(2n-3)\ldots 3\times 1 \qquad (10)$$

Putting this result into 2 we get

$$\left\langle x^{2n}\right\rangle = \frac{(2n-1)!!}{a^n} \tag{11}$$

We can also get this result from the variant Gaussian integral, obtained by completing the square in the exponent:

$$\int_{-\infty}^{\infty} dx e^{-ax^2/2+Jx} = \sqrt{\frac{2\pi}{a}} e^{J^2/2a} \tag{12}$$

If we take the derivative of the LHS with respect to J we get

$$\frac{d^{2n}}{dJ^{2n}} \int_{-\infty}^{\infty} dx e^{-ax^2/2+Jx} = \int_{-\infty}^{\infty} dx e^{-ax^2/2+Jx} x^{2n} \tag{13}$$

Setting $J=0$ and comparing with 2 gives us

$$\left\langle x^{2n}\right\rangle = \left.\frac{d^{2n}}{dJ^{2n}} e^{J^2/2a}\right|_{J=0} \tag{14}$$

Although I could leave things here, I decided that it would be interesting to prove that the RHS actually does give 11. This proved to be a bit trickier than I expected, but the derivation is interesting so I'll give it here.

We need to find a general expression for $\frac{d^{2n}}{dJ^{2n}} e^{J^2/2a}$ before setting $J=0$. To help with this, we can write out the derivative explicitly for the first few values of n. It's actually easier to do the work for $a=1$; using dimensional analysis it's easy enough to put a back in at the end. We find

n	$\frac{d^{2n}}{dJ^{2n}} e^{J^2/2}$
1	$e^{J^2/2}\left[1+J^2\right]$
2	$e^{J^2/2}\left[3+6J^2+J^4\right]$
3	$e^{J^2/2}\left[15+45J^2+15J^4+J^6\right]$
4	$e^{J^2/2}\left[105+420J^2+210J^4+28J^6+J^8\right]$
5	$e^{J^2/2}\left[945+4725J^2+3150J^4+630J^6+45J^8+J^{10}\right]$

We see that the constant term in each case is indeed equal to $(2n-1)!!$. The coefficient of the second highest power of J is $(2n-1)n$. Working backwards in each line, we find that the coefficient of the third highest power is $\frac{1}{2}(2n-1)(2n-3)n(n-1)$, of the fourth highest power is $\frac{1}{3\times 2}(2n-1)(2n-3)(2n-5)n(n-1)(n-2)$ and so on. In general, the coefficient of J^{2n-2m} is

$$\frac{n!}{m!(n-m)!}\frac{(2n-1)!!}{(2n-2m-1)!!} = \binom{n}{m}\frac{(2n-1)!!}{(2n-2m-1)!!} \tag{15}$$

Therefore we propose that

$$\frac{d^{2n}}{dJ^{2n}}e^{J^2/2}=e^{J^2/2}\sum_{m=0}^{n}\binom{n}{m}\frac{(2n-1)!!}{(2n-2m-1)!!}J^{2n-2m} \tag{16}$$

We can prove this in general using induction. We've already established the anchor step, since this formula is true for $n=1\ldots5$, so we can assume 16 for some value n and then work from there to prove it's true for $n+1$. That is, we want to show, starting from 16, that the following is true:

$$\frac{d^{2n+2}}{dJ^{2n+2}}e^{J^2/2}=e^{J^2/2}\sum_{m=0}^{n+1}\binom{n+1}{m}\frac{(2n+1)!!}{(2n-2m+1)!!}J^{2n-2m+2} \tag{17}$$

We need to take the derivative of 16 twice. We get

$$\frac{d^{2n+1}}{dJ^{2n+1}}e^{J^2/2}=e^{J^2/2}\sum_{m=0}^{n}\binom{n}{m}\frac{(2n-1)!!}{(2n-2m-1)!!}\left[J^{2n-2m+1}+(2n-2m)J^{2n-2m-1}\right] \tag{18}$$

$$\begin{aligned}\frac{d^{2n+2}}{dJ^{2n+2}}e^{J^2/2}=e^{J^2/2}&\sum_{m=0}^{n}\binom{n}{m}\frac{(2n-1)!!}{(2n-2m-1)!!}\times\\&\left[J^{2n-2m+2}+(4n-4m+1)J^{2n-2m}+(2n-2m)(2n-2m-1)J^{2n-2m-2}\right]\end{aligned} \tag{19}$$

We'd like to put this in the form 17, so we can shift the summation index from $m\rightarrow m-1$ in the second term, and from $m\rightarrow m-2$ in the third term, thus allowing us to factor out $J^{2n-2m+2}$ from all 3 terms. The limits on the sums will also change, so the second term now has limits of $m=1\ldots n+1$ and the third term of $m=2\ldots n+2$. We get (putting the exponential on the LHS for convenience):

$$\begin{aligned}e^{-J^2/2}\frac{d^{2n+2}}{dJ^{2n+2}}e^{J^2/2}=&\sum_{m=0}^{n}\binom{n}{m}\frac{(2n-1)!!}{(2n-2m-1)!!}J^{2n-2m+2}+\\&\sum_{m=1}^{n+1}\binom{n}{m-1}\frac{(2n-1)!!}{(2n-2m+1)!!}(4n-4m+5)J^{2n-2m+2}+\\&\sum_{m=2}^{n+2}\binom{n}{m-2}\frac{(2n-1)!!}{(2n-2m+3)!!}(2n-2m+4)(2n-2m+3)J^{2n-2m+2}\end{aligned} \tag{20}$$

We can condense the sums by using the fact that the binomial coefficient $\binom{p}{q}$ is zero if $q<0$ or $q>p$, so we can extend the lower limits on the second and third sums to 0, and extend the upper limit on the first sum to $n+1$. Also, in the third sum, the factor $(2n-2m+4)$ is zero when $m=n+2$, so we can reduce the upper limit on the sum to $n+1$. Therefore all three sums extend from $m=0\ldots n+1$ and we have (cancelling the common factor in the third sum as well):

$$\begin{aligned} e^{-J^2/2}\frac{d^{2n+2}}{dJ^{2n+2}}e^{J^2/2} = & \sum_{m=0}^{n+1}\binom{n}{m}\frac{(2n-1)!!}{(2n-2m-1)!!}J^{2n-2m+2}+ \\ & \sum_{m=0}^{n+1}\binom{n}{m-1}\frac{(2n-1)!!}{(2n-2m+1)!!}(4n-4m+5)J^{2n-2m+2}+ \\ & \sum_{m=0}^{n+1}\binom{n}{m-2}\frac{(2n-1)!!}{(2n-2m+1)!!}(2n-2m+4)J^{2n-2m+2} \end{aligned} \tag{21}$$

To convert the binomial coefficients, we have

$$\binom{n}{m-2} = \frac{m-1}{n-m+2}\binom{n}{m-1} \tag{22}$$

This allows us to combine the second and third sums:

$$\begin{aligned} e^{-J^2/2}\frac{d^{2n+2}}{dJ^{2n+2}}e^{J^2/2} = & \sum_{m=0}^{n+1}\binom{n}{m}\frac{(2n-1)!!}{(2n-2m-1)!!}J^{2n-2m+2}+ \\ & \sum_{m=0}^{n+1}\binom{n}{m-1}\frac{(2n-1)!!}{(2n-2m+1)!!}(4n-2m+3)J^{2n-2m+2} \end{aligned} \tag{23}$$

Also

$$\binom{n}{m-1} = \frac{m}{n-m+1}\binom{n}{m} \tag{24}$$

Using this, and multiplying the first sum by $\frac{2n-2m+1}{2n-2m+1}$ we can combine it with the second sum to get

$$e^{-J^2/2}\frac{d^{2n+2}}{dJ^{2n+2}}e^{J^2/2} = \sum_{m=0}^{n+1}\binom{n}{m}\frac{(2n-1)!!}{(2n-2m+1)!!}\left(2n-2m+1+\frac{m(4n-2m+3)}{n-m+1}\right)J^{2n-2m+2} \tag{25}$$

Finally, we have

$$\binom{n}{m} = \frac{n-m+1}{n+1}\binom{n+1}{m} \tag{26}$$

Plugging this in and simplifying, we get

$$\left(2n-2m+1+\frac{m(4n-2m+3)}{n-m+1}\right)\binom{n}{m} = \left(2n-2m+1+\frac{m(4n-2m+3)}{n-m+1}\right)\frac{n-m+1}{n+1}\binom{n+1}{m} \tag{27}$$

$$= (2n+1)\binom{n+1}{m} \tag{28}$$

$$e^{-J^2/2}\frac{d^{2n+2}}{dJ^{2n+2}}e^{J^2/2} = \sum_{m=0}^{n+1}\binom{n+1}{m}\frac{(2n+1)!!}{(2n-2m+1)!!}J^{2n-2m+2} \tag{29}$$

QED.

To restore the factors of a, we observe that J^2 has the same dimensions as a (since an exponent must be dimensionless), so the derivative $\frac{d^{2n}}{dJ^{2n}}$ has the dimensions of a^{-n}. Therefore, the term involving J^{2n-2m} must be divided by a^{2n-m}. Thus:

$$\frac{d^{2n}}{dJ^{2n}}e^{J^2/2a} = e^{J^2/2a}\sum_{m=0}^{n}\binom{n}{m}\frac{(2n-1)!!}{(2n-2m-1)!!}\frac{J^{2n-2m}}{a^{2n-m}} \tag{30}$$

When $J=0$, this reduces to the $m=n$ term, which is (the double factorial $(-1)!!=1$, at least according to Maple):

$$\left.\frac{d^{2n}}{dJ^{2n}}e^{J^2/2a}\right|_{J=0} = \frac{(2n-1)!!}{a^n} \tag{31}$$

which is the same as 11.

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