STEEPEST DESCENT AND THE CLASSICAL LIMIT OF QUANTUM MECHANICS

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References: Anthony Zee, *Quantum Field Theory in a Nutshell*, 2nd edition (Princeton University Press, 2010) - Chapter I.2, Appendix 3.

The Gaussian integral is a special type of exponential integral that happens to have an exact solution, in that, for some constant a:

$$\int_{-\infty}^{\infty} e^{-(x-a)^2} dx = \sqrt{\pi} \tag{1}$$

Suppose we have a more general integral of the form

$$I = \int_{-\infty}^{\infty} e^{-f(q)/\hbar} dq \tag{2}$$

where $f(q)$ is some arbitrary function that satisfies $f(q) \to \infty$ as $q \to \pm\infty$ (to ensure that I is finite). If $\hbar \to 0$ (OK, $\hbar$ is already pretty small, but we're looking at the case where it gets infinitesimally close to zero), then if $f(q)$ has only a single sharply defined minimum at $q = a$, the integral is essentially determined by the behaviour of f near the point $q = a$, since at that point $-f(q)/\hbar$ takes on its maximum value, maximizing the integrand. The idea of the method of steepest descent is that we can expand f about the point $q = a$ and use the first few terms in the expansion as an approximation in solving the integral.

Since $q = a$ is a minimum of $f(q)$, $f'(a) = 0$ and the Taylor expansion starts off as

$$f(q) = f(a) + \frac{1}{2} f''(a)(q-a)^2 + \mathcal{O}\left[(q-a)^3\right] \tag{3}$$

The integral then becomes

$$I = e^{-f(a)/\hbar} \int_{-\infty}^{\infty} e^{-f''(a)(q-a)^2/2\hbar} e^{-\mathcal{O}\left[(q-a)^3/\hbar\right]} dq \tag{4}$$

If we assume that the exponent in the first term in the integrand has order 1, then $\mathcal{O}\left[(q-a)^2\right] = \mathcal{O}(\hbar)$, that is, the the function $f(q)$ is close to its minimum value $f(a)$ only in a region around a such that $\mathcal{O}(q-a) = \mathcal{O}\left(\sqrt{\hbar}\right)$. Outside this region, the function is much larger and thus contributes a negligible amount to the integral I. Under this assumption

$$e^{-\mathcal{O}\left[(q-a)^3/\hbar\right]}=e^{-\mathcal{O}\left(\sqrt{\hbar}\right)} \tag{5}$$

and we can take it outside the integral, leaving

$$I = e^{-f(a)/\hbar}e^{-\mathcal{O}\left(\sqrt{\hbar}\right)}\int_{-\infty}^{\infty}e^{-f''(a)(q-a)^2/2\hbar}dq \tag{6}$$

$$= e^{-f(a)/\hbar}e^{-\mathcal{O}\left(\sqrt{\hbar}\right)}\left(\frac{2\pi\hbar}{f''(a)}\right)^{1/2} \tag{7}$$

In the more general case where f is a function of many variables q_i and has a sharp minimum at the N-dimensional point $q_i=a_i$ for $i=1,\ldots,N$, the Taylor expansion is

$$f(q)=f(a)+\frac{1}{2}\sum_{i,j}(q_i-a_i)(q_j-a_j)\left.\frac{\partial^2 f}{\partial q_i\partial q_j}\right|_{q=a}+\mathcal{O}(q-a)^3 \tag{8}$$

This can be written in matrix notation if we define

$$x_i \equiv q_i-a_i \tag{9}$$

$$A_{ij} = \left.\frac{\partial^2 f}{\partial q_i\partial q_j}\right|_{q=a} \tag{10}$$

Then we have, using the result in the previous post

$$f(q)=f(a)+x^TAx+\mathcal{O}(q-a)^3 \tag{11}$$

$$I = e^{-f(a)/\hbar}e^{-\mathcal{O}\left(\sqrt{\hbar}\right)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}dx_1dx_2\ldots dx_N e^{-\frac{1}{2}x^TAx} \tag{12}$$

$$= e^{-f(a)/\hbar}e^{-\mathcal{O}\left(\sqrt{\hbar}\right)}\left(\frac{(2\pi\hbar)^N}{\det A}\right) \tag{13}$$

One application of the steepest descent method is deriving the classical limit of quantum mechanics expressed in the path integral form. That is, we start from the path integral

$$\left\langle q_F\left|e^{-iHT}\right|q_I\right\rangle=\int Dq(t)\,e^{i\int_0^T\left[\frac{1}{2}m\dot{q}^2-V(q)\right]dt} \tag{14}$$

where the path integral operator is defined as

$$\int Dq(t)\equiv\lim_{N\to\infty}\left(\frac{-im}{2\pi\delta t}\right)^{N/2}\left[\prod_{j=1}^{N-1}\int dq_j\right] \tag{15}$$

These formulas are written in natural units with $\hbar = 1$, but to derive the classical limit we need to let $\hbar \to 0$ so we'll need to restore it in the formulas. The integral $\int_0^T \left[\frac{1}{2}m\dot{q}^2 - V(q)\right]$ in the exponent has the units of energy $\times$ time, which are the units of action, which are also the units of $\hbar$. To make the exponent dimensionless, we therefore divide it by $\hbar$ to get

$$\left\langle q_F \left| e^{-iHT} \right| q_I \right\rangle = \int Dq(t)\, e^{i\int_0^T \left[\frac{1}{2}m\dot{q}^2 - V(q)\right]dt/\hbar} \tag{16}$$

We can define the action as

$$S \equiv \int_0^T \left[\frac{1}{2}m\dot{q}^2 - V(q)\right] dt \tag{17}$$

Now if we let $\hbar \to 0$, we can apply the steepest descent method. [Actually, since the exponent is purely imaginary in this case, we can't use the argument that the exponential goes to zero when the exponent is large and negative. The argument here appears to be a bit more subtle, although Zee ignores this point. Since the path integral 15 integrates over all possible paths, I imagine the argument is that when the action S is large, slight changes in the path cause rapid oscillations in the exponential which tend to cancel each other out. It is only when S is near its minimum that slight changes in the paths do not cancel.]

That is, the method of steepest descent gives us the principle of least action, that is

$$\left\langle q_F \left| e^{-iHT} \right| q_I \right\rangle \to e^{i\int_0^T \left[\frac{1}{2}m\dot{q}_c{}^2 - V(q_c)\right]dt/\hbar} \tag{18}$$

where q_c is the path followed by a classical particle, and is determined by solving the Euler-Lagrange equations from classical mechanics

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0 \tag{19}$$

where the Lagrangian is defined as

$$L \equiv \frac{1}{2}m\dot{q}^2 - V(q) \tag{20}$$

Because only the path corresponding to the least action contributes to the path integral, the integral 15 over all possible paths collapses down to the single classical path q_c.