FROM MATTRESS TO FIELD

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

References: Anthony Zee, *Quantum Field Theory in a Nutshell*, 2nd edition (Princeton University Press, 2010) - Chapter I.3.

A D Boozer, *Quantum field theory in (0 + 1) dimensions*, Eur. J. Phys, **28** (2007) 729-745.

[Disclaimer: As I'm just starting out in learning quantum field theory, these notes are my attempts to make sense of Zee's textbook and should not be taken as authoritative on their own. Anyone who knows more QFT than I do is welcome to leave a comment correcting my interpretations.]

To get an idea of how a field theory is formed, Zee starts out with a 2-d mattress composed of a number point masses m_a placed at the intersection points of a square lattice, where each square has a side length ℓ. The masses are connected along the sides of all the squares by springs, where the spring connecting masses m_a and m_b has spring constant k_{ab}. If we jump up and down on the mattress, the masses will move up and down around their equilibrium points (we'll neglect any horizontal motion), with mass m_a having a vertical displacement $q_a(t)$ at time t. The kinetic energy is therefore

$$K=\sum_a \frac{1}{2}m_a\dot{q}_a^2 \tag{1}$$

To get the potential energy, we'll consider only nearest neighbour interactions, so the potential energy due to the spring connecting m_a and m_b is $\frac{1}{2}k_{ab}(q_a-q_b)^2$ and the total potential energy is

$$V=\sum_{a,b}\frac{1}{2}k_{ab}(q_a-q_b)^2 \tag{2}$$

Now suppose that the lattice spacing ℓ becomes infinitestimal. In this case we can define a surface mass density as the mass per unit area:

$$\sigma\equiv\frac{m_a}{\ell^2} \tag{3}$$

The displacement q_a now refers to a particular location $\vec{x}=(x_1,x_2)$ in the plane of the mattress rather than a particular lattice point, so we can write it as a continuous function of time and horizontal position:

$$q_a(t) \to \phi(t, \vec{x}) \tag{4}$$

The mass of the area enclosed by the infinitesimal rectangle $dx_1 dx_2$ is $\sigma dx_1 dx_2$ and the kinetic energy now becomes

$$K = \frac{1}{2} \int \int dx_1 dx_2 \sigma \left(\frac{\partial \phi}{\partial t}\right)^2 \tag{5}$$

In the most general case, σ would depend on position, but in this case we'll assume it's constant.

To work out the potential energy, we have

$$\begin{aligned} q_a - q_b &\to \phi(\vec{x}_a) - \phi(\vec{x}_b) && (6) \\ &\approx \ell \frac{\partial \phi}{\partial x} && (7) \end{aligned}$$

where the derivative is taken along the direction separating a and b. In the continuum limit, we therefore have (assuming only nearest neighbour interactions):

$$V = \frac{1}{2} \int \int dx_1 dx_2 \rho \left[\left(\frac{\partial \phi}{\partial x_1}\right)^2 + \left(\frac{\partial \phi}{\partial x_2}\right)^2\right] \tag{8}$$

where ρ depends on the spring constants k_{ab} and may depend on position within the plane.

In the path integral formulation of quantum mechanics, the amplitude for a system to go from some initial state I to a final state F is

$$\left\langle q_F \left| e^{-iHT} \right| q_I \right\rangle = \int Dq(t) e^{iSdt} \tag{9}$$

where S is the action, which is the integral of the Lagrangian L over time:

$$\begin{aligned} S &= \int_0^T L dt && (10) \\ &= \int_0^T (K - V) dt && (11) \end{aligned}$$

Thus for our continuum mattress, we have

$$S=\frac{1}{2}\int_{0}^{T}dt\int\int dx_{1}dx_{2}\left\{\sigma\left(\frac{\partial\phi}{\partial t}\right)^{2}-\rho\left[\left(\frac{\partial\phi}{\partial x_{1}}\right)^{2}+\left(\frac{\partial\phi}{\partial x_{2}}\right)^{2}\right]\right\} \tag{12}$$

We can scale things by introducing a velocity c and writing $\rho=\sigma c^{2}$. [The units are consistent since ϕ has the units of length and σ has the units of mass per unit area, so each term inside the braces has units of mass $\cdot$ time^{-2}, giving the action units of energy $\times$ time as required.] If we now scale $\phi\rightarrow\phi/\sqrt{\sigma}$ we get

$$S=\frac{1}{2}\int_{0}^{T}dt\int\int dx_{1}dx_{2}\left\{\left(\frac{\partial\phi}{\partial t}\right)^{2}-c^{2}\left[\left(\frac{\partial\phi}{\partial x_{1}}\right)^{2}+\left(\frac{\partial\phi}{\partial x_{2}}\right)^{2}\right]\right\} \tag{13}$$

The contents of the braces can be written in special relativistic notation as (taking $c=1$):

$$\partial_{\mu}\partial^{\mu}\phi=\left(\frac{\partial\phi}{\partial t}\right)^{2}-\left[\left(\frac{\partial\phi}{\partial x_{1}}\right)^{2}+\left(\frac{\partial\phi}{\partial x_{2}}\right)^{2}\right] \tag{14}$$

which transforms under Lorentz transformations as a scalar, so this quantity is Lorentz-invariant.

At this point, we need to step back and get a feel for what has happened. We've replaced the displacement q_a of a particular mass m_a by the displacement $\phi(\vec{x})$ of an element of mass at location $\vec{x}$. The function ϕ is a *field variable.* The position variable $\vec{x}$ isn't the dynamical position operator used in quantum mechanics; rather it is merely a label specifying which ϕ is being considered. In the mattress example above, the subscript a labelled which mass we were talking about; we could write an equation of motion for each mass. This equation of motion doesn't depend on the label a; we could change a without changing how the mass moves. In the continuum case, it is ϕ that determines how the element of mass at position $\vec{x}$ moves. The actual value of $\vec{x}$ is just a label that we use to locate the element of mass we're talking about.

The mattress example uses 2 space dimensions plus 1 time dimension, and can be referred to as a (2+1) dimensional spacetime. Models in d dimensions are referred to as $(D+1)$ dimensional, with D space dimensions and one time dimension.

Finally, it's worth looking at one of those annoying throw-away comments that Zee makes without explaining them. He states that a $(0+1)$ dimensional quantum field theory is equivalent to quantum mechanics. What does this mean? The paper by Boozer referred to at the top explains this as follows.

A scalar field ϕ in $(n+1)$ dimensions specifies a field value ϕ for every point in spacetime, that is

$$\phi = \phi\left(t, x_1, x_2, \ldots, x_n\right) \tag{15}$$

Here, the vector $\vec{x} = (x_1, x_2, \ldots, x_n)$ serves as a label for the particular field variable ϕ we're talking about, and this field variable evolves in time. In a $(0+1)$ dimensional QFT, there is only one field, which depends only on the time: $\phi = \phi\left(t\right)$. No spatial coordinates are involved.

In ordinary quantum mechanics in one spatial dimension, the particle's position can be given as a function of time: $x = x(t)$, (where I assume this means that x should be interpreted as the expectation value $\langle x\rangle$ since we're dealing with non-deterministic quantum mechanics). Both these cases use a mapping from the independent time variable t into a function (ϕ in QFT, x in quantum mechanics) determined by the theory. Precisely why this makes the two theories equivalent, however, is not clear to me. Comments?