

NOETHER'S THEOREM

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References: Anthony Zee, *Quantum Field Theory in a Nutshell*, 2nd edition (Princeton University Press, 2010) - Chapter I.10.

Michael E. Peskin & Daniel V. Schroeder, *An Introduction to Quantum Field Theory*, (Perseus Books, 1995) - Chapter 2.

We've seen that for the Klein-Gordon field, there is a probability current that is conserved. It is defined by

$$j_\mu = i \left(\phi^\dagger \partial_\mu \phi - \phi \partial_\mu \phi^\dagger \right) \quad (1)$$

and the conservation is expressed by its divergence being zero:

$$\partial^\mu j_\mu = 0 \quad (2)$$

This is a special case of a more general principle known as *Noether's theorem*. The idea is that if the Lagrangian density \mathcal{L} has a continuous symmetry, there is a corresponding current which is conserved.

First, what do we mean by a 'continuous symmetry'? If we can vary the field ϕ in some continuous manner (so that infinitesimal changes are possible) but in doing so, leave \mathcal{L} unchanged, then the Lagrangian has a continuous symmetry. In the more familiar language of 2-d geometry, for example, rotation of the coordinate system is a continuous operation, since we can rotate the coordinates by infinitesimal amounts. However, reflection of the coordinates about, say, the y axis, is *not* continuous since reflection is an all-or-nothing operation; there is no infinitesimal reflection.

Let's suppose we have a general Lagrangian of the form $\mathcal{L}(\phi_a, \partial_\mu \phi_a)$ where μ indicates a spacetime coordinate and a enumerates the fields, as usual. Now suppose this Lagrangian is invariant when we change the fields by some infinitesimal amounts $\delta\phi_a$. This means that $\delta\mathcal{L} = 0$ so we can say that

$$\delta\mathcal{L} = \frac{\delta\mathcal{L}}{\delta\phi_a} \delta\phi_a + \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_a} \delta\partial_\mu\phi_a \quad (3)$$

Using the Euler-Lagrange equations for fields, which are

$$\frac{\delta\mathcal{L}}{\delta\phi_a} - \partial_\mu \left(\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_a} \right) = 0 \quad (4)$$

we can write 3 as

$$\delta\mathcal{L} = \partial_\mu \left(\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_a} \right) \delta\phi_a + \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_a} \delta\partial_\mu\phi_a \quad (5)$$

Using the product rule, we can combine the two terms on the RHS to get

$$\delta\mathcal{L} = \partial_\mu \left(\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_a} \delta\phi_a \right) = 0 \quad (6)$$

That is, we've found a quantity (in parentheses) whose divergence is zero, so we can define this as a conserved current

$$j^\mu \equiv \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_a} \delta\phi_a \quad (7)$$

This is Noether's theorem.

Example 1. As an example, we can use the Lagrangian for the Klein-Gordon equation

$$\mathcal{L}_0 = \partial_\mu\phi^\dagger\partial^\mu\phi - \mu^2\phi^\dagger\phi \quad (8)$$

Treating ϕ and ϕ^\dagger as the two independent fields we note that since these two fields always occur as a product, the Lagrangian is unchanged if we replace

$$\phi \rightarrow e^{i\theta}\phi; \quad \phi^\dagger \rightarrow e^{-i\theta}\phi^\dagger \quad (9)$$

This is a continuous symmetry, since the parameter θ is a continuous variable. In infinitesimal form

$$\phi \rightarrow \phi + i\theta\phi \quad (10)$$

$$\delta\phi = i\theta\phi \quad (11)$$

$$\phi^\dagger \rightarrow \phi^\dagger - i\theta\phi^\dagger \quad (12)$$

$$\delta\phi^\dagger = -i\theta\phi^\dagger \quad (13)$$

We get

$$\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi} = \partial^\mu\phi^\dagger \quad (14)$$

$$\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^\dagger} = \partial^\mu\phi \quad (15)$$

Therefore, from 7

$$j^\mu = i\theta\phi\partial^\mu\phi^\dagger - i\theta\phi^\dagger\partial^\mu\phi \quad (16)$$

$$= -i\theta\left(\phi^\dagger\partial_\mu\phi - \phi\partial_\mu\phi^\dagger\right) \quad (17)$$

which apart from the $-\theta$ (which drops out when taking the divergence and setting to zero) is the same current we had earlier in 1.

If you refer back to the derivation of the Euler-Lagrange equations, you'll see that one of terms in the variation of the action was

$$\int_{\Omega} \frac{\partial}{\partial q^\mu} \left[\frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} \delta\phi \right] d^4q \quad (18)$$

where Ω is the volume over which the integration is done. The argument made there was that since the integrand is a divergence, we can use Gauss's theorem to convert this to a surface integral and since we're holding the fields constant on the boundary, $\delta\phi = 0$ on the boundary, so this integral is zero. Using the same argument, if we add a divergence term, say $\partial_\mu K^\mu$, to $\delta\mathcal{L}$, then the integral of this term over the volume is also zero, provided that $K^\mu = 0$ on the boundary. That is, we can replace 6 by

$$\delta\mathcal{L} = \partial_\mu \left(\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_a} \delta\phi_a - K^\mu \right) = 0 \quad (19)$$

so that the conserved current then becomes

$$j^\mu \equiv \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_a} \delta\phi_a - K^\mu \quad (20)$$

[The derivation of this current in Peskin & Schroeder - their equation 2.12 - is a bit muddled. They refer to K^μ as \mathcal{J}^μ and in the sentence before equation 2.12 they seem to state that

$$\partial_\mu \mathcal{J}^\mu = \partial_\mu \left(\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_a} \delta\phi_a \right) \quad (21)$$

and then state equation 20, which would of course imply that $j^\mu = 0$. The derivation in this post follows that given in Zee and is a lot clearer.]

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