NOETHER'S THEOREM

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References: Anthony Zee, *Quantum Field Theory in a Nutshell*, 2nd edition (Princeton University Press, 2010) - Chapter I.10.

Michael E. Peskin & Daniel V. Schroeder, *An Introduction to Quantum Field Theory*, (Perseus Books, 1995) - Chapter 2.

We've seen that for the Klein-Gordon field, there is a probability current that is conserved. It is defined by

$$j_\mu = i\left(\phi^\dagger \partial_\mu \phi - \phi \partial_\mu \phi^\dagger\right) \tag{1}$$

and the conservation is expressed by its divergence being zero:

$$\partial^\mu j_\mu = 0 \tag{2}$$

This is a special case of a more general principle known as *Noether's theorem*. The idea is that if the Lagrangian density $\mathcal{L}$ has a continuous symmetry, there is a corresponding current which is conserved.

First, what do we mean by a 'continuous symmetry'? If we can vary the field ϕ in some continuous manner (so that infinitesimal changes are possible) but in doing so, leave $\mathcal{L}$ unchanged, then the Lagrangian has a continuous symmetry. In the more familiar language of 2-d geometry, for example, rotation of the coordinate system is a continuous operation, since we can rotate the coordinates by infinitesimal amounts. However, reflection of the coordinates about, say, the y axis, is *not* continuous since reflection is an all-or-nothing operation; there is no infinitesimal reflection.

Let's suppose we have a general Lagrangian of the form $\mathcal{L}(\phi_a, \partial_\mu \phi_a)$ where μ indicates a spacetime coordinate and a enumerates the fields, as usual. Now suppose this Lagrangian is invariant when we change the fields by some infinitesimal amounts $\delta\phi_a$. This means that $\delta\mathcal{L} = 0$ so we can say that

$$\delta\mathcal{L} = \frac{\delta\mathcal{L}}{\delta\phi_a}\delta\phi_a + \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_a}\delta\partial_\mu\phi_a \tag{3}$$

Using the Euler-Lagrange equations for fields, which are

$$\frac{\delta\mathcal{L}}{\delta\phi_a} - \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_a}\right) = 0 \tag{4}$$

we can write 3 as

$$\delta\mathcal{L} = \partial_\mu \left(\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_a} \right) \delta\phi_a + \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_a} \delta\partial_\mu\phi_a \tag{5}$$

Using the product rule, we can combine the two terms on the RHS to get

$$\delta\mathcal{L} = \partial_\mu \left(\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_a} \delta\phi_a \right) = 0 \tag{6}$$

That is, we've found a quantity (in parentheses) whose divergence is zero, so we can define this as a conserved current

$$j^\mu \equiv \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_a} \delta\phi_a \tag{7}$$

This is Noether's theorem.

Example 1. As an example, we can use the Lagrangian for the Klein-Gordon equation

$$\mathcal{L}_0^0 = \partial_\mu\phi^\dagger\partial^\mu\phi - \mu^2\phi^\dagger\phi \tag{8}$$

Treating ϕ and $\phi^\dagger$ as the two independent fields we note that since these two fields always occur as a product, the Lagrangian is unchanged if we replace

$$\phi \to e^{i\theta}\phi; \;\; \phi^\dagger \to e^{-i\theta}\phi^\dagger \tag{9}$$

This is a continuous symmetry, since the parameter θ is a continuous variable. In infinitesimal form

$$\begin{aligned} \phi &\to \phi + i\theta\phi && (10) \\ \delta\phi &= i\theta\phi && (11) \\ \phi^\dagger &\to \phi^\dagger - i\theta\phi^\dagger && (12) \\ \delta\phi^\dagger &= -i\theta\phi^\dagger && (13) \end{aligned}$$

We get

$$\begin{aligned} \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi} &= \partial^\mu\phi^\dagger && (14) \\ \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^\dagger} &= \partial^\mu\phi && (15) \end{aligned}$$

Therefore, from 7

$$j^{\mu} = i\theta\phi\partial^{\mu}\phi^{\dagger} - i\theta\phi^{\dagger}\partial^{\mu}\phi \tag{16}$$
$$= -i\theta\left(\phi^{\dagger}\partial_{\mu}\phi - \phi\partial_{\mu}\phi^{\dagger}\right) \tag{17}$$

which apart from the $-\theta$ (which drops out when taking the divergence and setting to zero) is the same current we had earlier in 1.

If you refer back to the derivation of the Euler-Lagrange equations, you'll see that one of terms in the variation of the action was

$$\int_{\Omega}\frac{\partial}{\partial q^{\mu}}\left[\frac{\partial\mathcal{L}}{\partial\phi_{,\mu}}\delta\phi\right]d^4q \tag{18}$$

where Ω is the volume over which the integration is done. The argument made there was that since the integrand is a divergence, we can use Gauss's theorem to convert this to a surface integral and since we're holding the fields constant on the boundary, $\delta\phi = 0$ on the boundary, so this integral is zero. Using the same argument, if we add a divergence term, say $\partial_{\mu}K^{\mu}$, to $\delta\mathcal{L}$, then the integral of this term over the volume is also zero, provided that $K^{\mu} = 0$ on the boundary. That is, we can replace 6 by

$$\delta\mathcal{L} = \partial_{\mu}\left(\frac{\delta\mathcal{L}}{\delta\partial_{\mu}\phi_a}\delta\phi_a - K^{\mu}\right) = 0 \tag{19}$$

so that the conserved current then becomes

$$j^{\mu} \equiv \frac{\delta\mathcal{L}}{\delta\partial_{\mu}\phi_a}\delta\phi_a - K^{\mu} \tag{20}$$

[The derivation of this current in Peskin & Schroeder - their equation 2.12 - is a bit muddled. They refer to K^{μ} as $\mathcal{J}^{\mu}$ and in the sentence before equation 2.12 they seem to state that

$$\partial_{\mu}\mathcal{J}^{\mu} = \partial_{\mu}\left(\frac{\delta\mathcal{L}}{\delta\partial_{\mu}\phi_a}\delta\phi_a\right) \tag{21}$$

and then state equation 20, which would of course imply that $j^{\mu} = 0$. The derivation in this post follows that given in Zee and is a lot clearer.]

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