

## NOETHER'S THEOREM

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References: Anthony Zee, *Quantum Field Theory in a Nutshell*, 2nd edition (Princeton University Press, 2010) - Chapter I.10.

Michael E. Peskin & Daniel V. Schroeder, *An Introduction to Quantum Field Theory*, (Perseus Books, 1995) - Chapter 2.

We've seen that for the Klein-Gordon field, there is a probability current that is conserved. It is defined by

$$(1) \quad j_\mu = i \left( \phi^\dagger \partial_\mu \phi - \phi \partial_\mu \phi^\dagger \right)$$

and the conservation is expressed by its divergence being zero:

$$(2) \quad \partial^\mu j_\mu = 0$$

This is a special case of a more general principle known as *Noether's theorem*. The idea is that if the Lagrangian density  $\mathcal{L}$  has a continuous symmetry, there is a corresponding current which is conserved.

First, what do we mean by a 'continuous symmetry'? If we can vary the field  $\phi$  in some continuous manner (so that infinitesimal changes are possible) but in doing so, leave  $\mathcal{L}$  unchanged, then the Lagrangian has a continuous symmetry. In the more familiar language of 2-d geometry, for example, rotation of the coordinate system is a continuous operation, since we can rotate the coordinates by infinitesimal amounts. However, reflection of the coordinates about, say, the  $y$  axis, is *not* continuous since reflection is an all-or-nothing operation; there is no infinitesimal reflection.

Let's suppose we have a general Lagrangian of the form  $\mathcal{L}(\phi_a, \partial_\mu \phi_a)$  where  $\mu$  indicates a spacetime coordinate and  $a$  enumerates the fields, as usual. Now suppose this Lagrangian is invariant when we change the fields by some infinitesimal amounts  $\delta\phi_a$ . This means that  $\delta\mathcal{L} = 0$  so we can say that

$$(3) \quad \delta\mathcal{L} = \frac{\delta\mathcal{L}}{\delta\phi_a} \delta\phi_a + \frac{\delta\mathcal{L}}{\delta\partial_\mu \phi_a} \delta\partial_\mu \phi_a$$

Using the Euler-Lagrange equations for fields, which are

$$(4) \quad \frac{\delta \mathcal{L}}{\delta \phi_a} - \partial_\mu \left( \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi_a} \right) = 0$$

we can write 3 as

$$(5) \quad \delta \mathcal{L} = \partial_\mu \left( \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi_a} \right) \delta \phi_a + \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi_a} \delta \partial_\mu \phi_a$$

Using the product rule, we can combine the two terms on the RHS to get

$$(6) \quad \delta \mathcal{L} = \partial_\mu \left( \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi_a} \delta \phi_a \right) = 0$$

That is, we've found a quantity (in parentheses) whose divergence is zero, so we can define this as a conserved current

$$(7) \quad j^\mu \equiv \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi_a} \delta \phi_a$$

This is Noether's theorem.

**Example 1.** As an example, we can use the Lagrangian for the Klein-Gordon equation

$$(8) \quad \mathcal{L}_0 = \partial_\mu \phi^\dagger \partial^\mu \phi - \mu^2 \phi^\dagger \phi$$

Treating  $\phi$  and  $\phi^\dagger$  as the two independent fields we note that since these two fields always occur as a product, the Lagrangian is unchanged if we replace

$$(9) \quad \phi \rightarrow e^{i\theta} \phi; \quad \phi^\dagger \rightarrow e^{-i\theta} \phi^\dagger$$

This is a continuous symmetry, since the parameter  $\theta$  is a continuous variable. In infinitesimal form

$$(10) \quad \phi \rightarrow \phi + i\theta \phi$$

$$(11) \quad \delta \phi = i\theta \phi$$

$$(12) \quad \phi^\dagger \rightarrow \phi^\dagger - i\theta \phi^\dagger$$

$$(13) \quad \delta \phi^\dagger = -i\theta \phi^\dagger$$

We get

$$(14) \quad \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi} = \partial^\mu \phi^\dagger$$

$$(15) \quad \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi^\dagger} = \partial^\mu \phi$$

Therefore, from 7

$$(16) \quad j^\mu = i\theta \phi \partial^\mu \phi^\dagger - i\theta \phi^\dagger \partial^\mu \phi$$

$$(17) \quad = -i\theta \left( \phi^\dagger \partial_\mu \phi - \phi \partial_\mu \phi^\dagger \right)$$

which apart from the  $-\theta$  (which drops out when taking the divergence and setting to zero) is the same current we had earlier in 1.

If you refer back to the derivation of the Euler-Lagrange equations, you'll see that one of terms in the variation of the action was

$$(18) \quad \int_\Omega \frac{\partial}{\partial q^\mu} \left[ \frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} \delta \phi \right] d^4 q$$

where  $\Omega$  is the volume over which the integration is done. The argument made there was that since the integrand is a divergence, we can use Gauss's theorem to convert this to a surface integral and since we're holding the fields constant on the boundary,  $\delta \phi = 0$  on the boundary, so this integral is zero. Using the same argument, if we add a divergence term, say  $\partial_\mu K^\mu$ , to  $\delta \mathcal{L}$ , then the integral of this term over the volume is also zero, provided that  $K^\mu = 0$  on the boundary. That is, we can replace 6 by

$$(19) \quad \delta \mathcal{L} = \partial_\mu \left( \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi_a} \delta \phi_a - K^\mu \right) = 0$$

so that the conserved current then becomes

$$(20) \quad j^\mu \equiv \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi_a} \delta \phi_a - K^\mu$$

[The derivation of this current in Peskin & Schroeder - their equation 2.12 - is a bit muddled. They refer to  $K^\mu$  as  $\mathcal{J}^\mu$  and in the sentence before equation 2.12 they seem to state that

$$(21) \quad \partial_\mu \mathcal{J}^\mu = \partial_\mu \left( \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi_a} \delta \phi_a \right)$$

and then state equation 20, which would of course imply that  $j^\mu = 0$ . The derivation in this post follows that given in Zee and is a lot clearer.]

#### PINGBACKS

Pingback: [Noether's theorem and conservation laws](#)