# GAUSSIAN INTEGRALS: AVERAGES OVER MATRIX COMPONENTS AND THE WICK CONTRACTION

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References: Anthony Zee, *Quantum Field Theory in a Nutshell*, 2nd edition (Princeton University Press, 2010) - Chapter I.2, Problem I.2.2.

We've seen how to evaluate a Gaussian integral with matrices in the exponent:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}dx_1dx_2\ldots dx_N e^{-\frac{1}{2}x^TAx+J^Tx}=\sqrt{\frac{(2\pi)^N}{\det A}}e^{\frac{1}{2}J^TA^{-1}J} \tag{1}$$

Using this formula, we can generalize the definition of averages of powers of $x$ in the single variable integral. That is, we would like to calculate

$$\left\langle x_ix_j\ldots x_kx_\ell\right\rangle\equiv\frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}dx_1dx_2\ldots dx_Nx_ix_j\ldots x_kx_\ell e^{-\frac{1}{2}x^TAx}}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}dx_1dx_2\ldots dx_Ne^{-\frac{1}{2}x^TAx}} \tag{2}$$

From the LHS of 1, we see that this average can be obtained by taking the derivative with respect to $J_a$ for each subscript $a$ in the set of $x_a$s that we want to average, and then setting $J=0$. For example, since $J^Tx=\sum_a x_aJ_a$,

$$\frac{\partial^2}{\partial J_i\partial J_j}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}dx_1dx_2\ldots dx_Ne^{-\frac{1}{2}x^TAx+J^Tx}= \tag{3}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}dx_1dx_2\ldots dx_Nx_ix_je^{-\frac{1}{2}x^TAx+J^Tx} \tag{4}$$

Therefore

$$\left\langle x_ix_j\right\rangle = \left.\frac{\frac{\partial^2}{\partial J_i\partial J_j}\left(\sqrt{\frac{(2\pi)^N}{\det A}}e^{\frac{1}{2}J^TA^{-1}J}\right)}{\sqrt{\frac{(2\pi)^N}{\det A}}e^{\frac{1}{2}J^TA^{-1}J}}\right|_{J=0} \tag{5}$$

$$= \left.\frac{\partial^2}{\partial J_i\partial J_j}\left(e^{\frac{1}{2}J^TA^{-1}J}\right)\right|_{J=0} \tag{6}$$

Working out these derivatives isn't all that bad, if we do the first few to see how the pattern goes. To make the notation a bit easier, we'll define the following:

$$\alpha \equiv e^{\frac{1}{2}J^T A^{-1} J} \tag{7}$$
$$a \equiv A^{-1} \tag{8}$$
$$\beta_i \equiv A^{-1}_{ik} J_k = a_{ik} J_k \tag{9}$$
$$\partial_i \equiv \frac{\partial}{\partial J_i} \tag{10}$$

with an implied sum over $k$ in the definition of $\beta$. Repeated indices within the same term are always summed in what follows.

Taking the first derivative, we get

$$\begin{aligned}
\partial_k \alpha &= \frac{\alpha}{2}\left(a_{kj}J_j + J_i a_{ik}\right) && (11)\\
&= \alpha a_{kj} J_j && (12)\\
&= \alpha \beta_k && (13)
\end{aligned}$$

where the second line follows because $A$ and therefore $A^{-1} = a$ are both symmetric matrices. Note that $J = 0$ implies $\beta = 0$ and $\alpha = 1$.

In subsequent derivatives, we'll need the result

$$\begin{aligned}
\partial_\ell \beta_k &= \frac{\partial}{\partial J_\ell} a_{ik} J_k && (14)\\
&= a_{i\ell} && (15)
\end{aligned}$$

The second derivative is, from 13

$$\begin{aligned}
\partial_\ell \partial_k \alpha &= \beta_k \partial_\ell \alpha + \alpha \partial_\ell \beta_k && (16)\\
&= \alpha \beta_k \beta_\ell + \alpha a_{\ell k} && (17)\\
&= a_{\ell k} = A^{-1}_{\ell k} \text{ (for } J = 0) && (18)
\end{aligned}$$

Therefore

$$\langle x_\ell x_k \rangle = A^{-1}_{\ell k} \tag{19}$$

The third derivative is

$$\partial_m\partial_\ell\partial_k\alpha = (\partial_m\alpha)\beta_k\beta_\ell + \alpha(\partial_m\beta_k)\beta_\ell + \alpha\beta_k(\partial_m\beta_\ell) + (\partial_m\alpha)a_{\ell k} \tag{20}$$
$$= \alpha\beta_m\beta_k\beta_\ell + \alpha a_{mk}\beta_\ell + \alpha\beta_k a_{m\ell} + \alpha\beta_m a_{\ell k} \tag{21}$$
$$= 0 \text{ (for } J = 0) \tag{22}$$

The fourth derivative is

$$\partial_n\partial_m\partial_\ell\partial_k\alpha = \alpha\beta_n\beta_m\beta_k\beta_\ell + \alpha\beta_n a_{mk}\beta_\ell + \alpha\beta_n\beta_k a_{m\ell} + \alpha\beta_n\beta_m a_{\ell k} +$$
$$\alpha a_{mk}a_{\ell n} + \alpha a_{kn}a_{m\ell} + \alpha a_{mn}a_{\ell k} \tag{23}$$
$$= a_{mk}a_{\ell n} + a_{kn}a_{m\ell} + a_{mn}a_{\ell k} \text{ (for } J = 0) \tag{24}$$

To see the general pattern for the derivative $\partial_i\partial_j\ldots\partial_k\partial_\ell$ containing $N$ factors, note that the first term is always $\alpha\beta_i\beta_j\ldots\beta_k\beta_\ell$, that is, it contains $\alpha$ multiplied by all $N$ possible $\beta_i$s. Then there is a set of terms consisting of $\alpha$ multiplied by $N-2$ $\beta_i$s and one $a_{ij}$. The number of these terms is equal to the number of unique permutations of the $N$ indices, allowing for the symmetry of $a_{ij}$. For example, in the fourth derivative above, there are 3 unique ways of distributing the 4 indices among a product of form $a_{ij}\beta_k\beta_\ell$, so there are 3 of these terms.

Next there are terms consisting of $\alpha$ multiplied by $N-4$ $\beta_i$s and 2 $a_{ij}$s. Again, the number of terms is equal to the number of unique permutations of the $N$ indices among the factors in each term, allowing for the symmetry of $a_{ij}$. In the fourth derivative, this gives terms containing zero $\beta_i$s and two $a_{ij}$s, and there are 3 unique ways of distributing 4 indices between the two $a_{ij}$s.

The process continues $n$ times, where $n$ is determined by the condition $N-2n=0$ (for even-order derivatives) or 1 (for odd-order derivatives). For odd-order derivatives, all terms contain at least one factor $\beta_i$, so all these derivatives are zero when $J=0$. For even-order derivatives, we get

$$\langle x_i x_j \ldots x_k x_\ell \rangle = \sum_{Wick} A^{-1}_{ab}\ldots A^{-1}_{cd} \tag{25}$$

where each term in the sum is a product of $\frac{N}{2}$ $A^{-1}_{ij}$ elements, and the sum is over all unique permutations of the $N$ indices distributed amongst these elements. This set of permutations is known as a *Wick contraction*.

For the case $N=1$, 2 reduces to the single-variable case we considered earlier, where we found that

$$\langle x^{2n} \rangle = \frac{(2n-1)!!}{a^n} \tag{26}$$

Since each term in the Wick sum contributes the same amount $\frac{1}{a^n}$ in this case, there are $(2n-1)!!$ terms in the sum.

With these rules, we can write down the sixth-order expansion ($N = 2n = 6$), for which there are $(2\times 3-1)!! = 15$ terms in the Wick sum:

$$\begin{aligned}\left\langle x_i x_j x_k x_\ell x_m x_n \right\rangle = {} & a_{ij}a_{k\ell}a_{mn} + a_{ij}a_{km}a_{\ell n} + a_{ij}a_{kn}a_{\ell m} + \\ & a_{ik}a_{j\ell}a_{mn} + a_{ik}a_{jm}a_{\ell n} + a_{ik}a_{jn}a_{\ell m} + \\ & a_{i\ell}a_{jk}a_{mn} + a_{i\ell}a_{jm}a_{kn} + a_{i\ell}a_{jn}a_{km} + \\ & a_{im}a_{jk}a_{\ell n} + a_{im}a_{j\ell}a_{kn} + a_{im}a_{jn}a_{k\ell} + \\ & a_{in}a_{jk}a_{\ell m} + a_{in}a_{j\ell}a_{km} + a_{in}a_{jm}a_{k\ell}\end{aligned} \tag{27}$$

The pattern followed pairs the first two indices $i$ and $j$ in $a_{ij}$, then works out the Wick contraction of the remaining four indices $k,\ell,m,n$ to produce the first 3 terms. Then pair $i$ with $k$ in $a_{ik}$ and work out the Wick contraction of the other four indices $j,\ell,m,n$ to get the next 3 terms and so on. From this process we can also derive the number of terms in a Wick sum of order $N = 2n$. For $N = 2$, there is only one permutation. For $N = 4$, we can pair the first index with any of the 3 remaining indices, leaving 2 indices which as we've just seen, have only one permutation. Thus for $N = 4$ the number of permutations is $3\times 1 = 3$. For $N = 6$, the first index can be paired with any of the 5 remaining indices, leaving 4 other indices which can be permuted in 3 ways, so the number of terms is $5\times 3\times 1 = 15$ and so on.