GAUSSIAN INTEGRALS: AVERAGES OVER MATRIX COMPONENTS AND THE WICK CONTRACTION

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References: Anthony Zee, *Quantum Field Theory in a Nutshell*, 2nd edition (Princeton University Press, 2010) - Chapter I.2, Problem I.2.2.

We've seen how to evaluate a Gaussian integral with matrices in the exponent:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}dx_1dx_2\ldots dx_N e^{-\frac{1}{2}x^TAx+J^Tx}=\sqrt{\frac{(2\pi)^N}{\det A}}e^{\frac{1}{2}J^TA^{-1}J} \tag{1}$$

Using this formula, we can generalize the definition of averages of powers of x in the single variable integral. That is, we would like to calculate

$$\left\langle x_ix_j\ldots x_kx_\ell\right\rangle \equiv \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}dx_1dx_2\ldots dx_Nx_ix_j\ldots x_kx_\ell e^{-\frac{1}{2}x^TAx}}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}dx_1dx_2\ldots dx_N e^{-\frac{1}{2}x^TAx}} \tag{2}$$

From the LHS of 1, we see that this average can be obtained by taking the derivative with respect to J_a for each subscript a in the set of x_as that we want to average, and then setting $J=0$. For example, since $J^Tx=\sum_a x_aJ_a$,

$$\frac{\partial^2}{\partial J_i\partial J_j}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}dx_1dx_2\ldots dx_N e^{-\frac{1}{2}x^TAx+J^Tx}= \tag{3}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}dx_1dx_2\ldots dx_Nx_ix_j e^{-\frac{1}{2}x^TAx+J^Tx} \tag{4}$$

Therefore

$$\left\langle x_ix_j\right\rangle = \left.\frac{\frac{\partial^2}{\partial J_i\partial J_j}\left(\sqrt{\frac{(2\pi)^N}{\det A}}e^{\frac{1}{2}J^TA^{-1}J}\right)}{\sqrt{\frac{(2\pi)^N}{\det A}}e^{\frac{1}{2}J^TA^{-1}J}}\right|_{J=0} \tag{5}$$

$$= \left.\frac{\partial^2}{\partial J_i\partial J_j}\left(e^{\frac{1}{2}J^TA^{-1}J}\right)\right|_{J=0} \tag{6}$$

Working out these derivatives isn't all that bad, if we do the first few to see how the pattern goes. To make the notation a bit easier, we'll define the following:

$$\alpha \equiv e^{\frac{1}{2}J^TA^{-1}J} \tag{7}$$
$$a \equiv A^{-1} \tag{8}$$
$$\beta_i \equiv A^{-1}_{ik}J_k = a_{ik}J_k \tag{9}$$
$$\partial_i \equiv \frac{\partial}{\partial J_i} \tag{10}$$

with an implied sum over k in the definition of β. Repeated indices within the same term are always summed in what follows.

Taking the first derivative, we get

$$\partial_k\alpha = \frac{\alpha}{2}\left(a_{kj}J_j + J_ia_{ik}\right) \tag{11}$$
$$= \alpha a_{kj}J_j \tag{12}$$
$$= \alpha\beta_k \tag{13}$$

where the second line follows because A and therefore $A^{-1} = a$ are both symmetric matrices. Note that $J = 0$ implies $\beta = 0$ and $\alpha = 1$.

In subsequent derivatives, we'll need the result

$$\partial_\ell\beta_k = \frac{\partial}{\partial J_\ell}a_{ik}J_k \tag{14}$$
$$= a_{i\ell} \tag{15}$$

The second derivative is, from 13

$$\partial_\ell\partial_k\alpha = \beta_k\partial_\ell\alpha + \alpha\partial_\ell\beta_k \tag{16}$$
$$= \alpha\beta_k\beta_\ell + \alpha a_{\ell k} \tag{17}$$
$$= a_{\ell k} = A^{-1}_{\ell k} \text{ (for } J = 0) \tag{18}$$

Therefore

$$\langle x_\ell x_k\rangle = A^{-1}_{\ell k} \tag{19}$$

The third derivative is

$$\begin{aligned}
\partial_m\partial_\ell\partial_k\alpha &= (\partial_m\alpha)\,\beta_k\beta_\ell + \alpha\,(\partial_m\beta_k)\,\beta_\ell + \alpha\beta_k\,(\partial_m\beta_\ell) + (\partial_m\alpha)\,a_{\ell k} && (20)\\
&= \alpha\beta_m\beta_k\beta_\ell + \alpha a_{mk}\beta_\ell + \alpha\beta_k a_{m\ell} + \alpha\beta_m a_{\ell k} && (21)\\
&= 0 \text{ (for } J=0) && (22)
\end{aligned}$$

The fourth derivative is

$$\begin{aligned}
\partial_n\partial_m\partial_\ell\partial_k\alpha = {} & \alpha\beta_n\beta_m\beta_k\beta_\ell + \alpha\beta_n a_{mk}\beta_\ell + \alpha\beta_n\beta_k a_{m\ell} + \alpha\beta_n\beta_m a_{\ell k} + \\
& \alpha a_{mk}a_{\ell n} + \alpha a_{kn}a_{m\ell} + \alpha a_{mn}a_{\ell k} && (23)\\
= {} & a_{mk}a_{\ell n} + a_{kn}a_{m\ell} + a_{mn}a_{\ell k} \text{ (for } J=0) && (24)
\end{aligned}$$

To see the general pattern for the derivative $\partial_i\partial_j\ldots\partial_k\partial_\ell$ containing N factors, note that the first term is always $\alpha\beta_i\beta_j\ldots\beta_k\beta_\ell$, that is, it contains α multiplied by all N possible β_is. Then there is a set of terms consisting of α multiplied by $N-2$ β_is and one a_{ij}. The number of these terms is equal to the number of unique permutations of the N indices, allowing for the symmetry of a_{ij}. For example, in the fourth derivative above, there are 3 unique ways of distributing the 4 indices among a product of form $a_{ij}\beta_k\beta_\ell$, so there are 3 of these terms.

Next there are terms consisting of α multiplied by $N-4$ β_is and 2 a_{ij}s. Again, the number of terms is equal to the number of unique permutations of the N indices among the factors in each term, allowing for the symmetry of a_{ij}. In the fourth derivative, this gives terms containing zero β_is and two a_{ij}s, and there are 3 unique ways of distributing 4 indices between the two a_{ij}s.

The process continues n times, where n is determined by the condition $N-2n=0$ (for even-order derivatives) or 1 (for odd-order derivatives). For odd-order derivatives, all terms contain at least one factor β_i, so all these derivatives are zero when $J=0$. For even-order derivatives, we get

$$\left\langle x_i x_j \ldots x_k x_\ell \right\rangle = \sum_{Wick} A^{-1}_{ab}\ldots A^{-1}_{cd} \qquad (25)$$

where each term in the sum is a product of $\frac{N}{2}$ A^{-1}_{ij} elements, and the sum is over all unique permutations of the N indices distributed amongst these elements. This set of permutations is known as a *Wick contraction*.

For the case $N=1$, 2 reduces to the single-variable case we considered earlier, where we found that

$$\left\langle x^{2n} \right\rangle = \frac{(2n-1)!!}{a^n} \qquad (26)$$

Since each term in the Wick sum contributes the same amount $\frac{1}{a^n}$ in this case, there are $(2n-1)!!$ terms in the sum.

With these rules, we can write down the sixth-order expansion ($N = 2n = 6$), for which there are $(2\times 3-1)!! = 15$ terms in the Wick sum:

$$\begin{aligned}\left\langle x_i x_j x_k x_\ell x_m x_n\right\rangle = {} & a_{ij}a_{k\ell}a_{mn} + a_{ij}a_{km}a_{\ell n} + a_{ij}a_{kn}a_{\ell m} + \\ & a_{ik}a_{j\ell}a_{mn} + a_{ik}a_{jm}a_{\ell n} + a_{ik}a_{jn}a_{\ell m} + \\ & a_{i\ell}a_{jk}a_{mn} + a_{i\ell}a_{jm}a_{kn} + a_{i\ell}a_{jn}a_{km} + \\ & a_{im}a_{jk}a_{\ell n} + a_{im}a_{j\ell}a_{kn} + a_{im}a_{jn}a_{k\ell} + \\ & a_{in}a_{jk}a_{\ell m} + a_{in}a_{j\ell}a_{km} + a_{in}a_{jm}a_{k\ell}\end{aligned} \tag{27}$$

The pattern followed pairs the first two indices i and j in a_{ij}, then works out the Wick contraction of the remaining four indices k,ℓ,m,n to produce the first 3 terms. Then pair i with k in a_{ik} and work out the Wick contraction of the other four indices j,ℓ,m,n to get the next 3 terms and so on. From this process we can also derive the number of terms in a Wick sum of order $N = 2n$. For $N = 2$, there is only one permutation. For $N = 4$, we can pair the first index with any of the 3 remaining indices, leaving 2 indices which as we've just seen, have only one permutation. Thus for $N = 4$ the number of permutations is $3\times 1 = 3$. For $N = 6$, the first index can be paired with any of the 5 remaining indices, leaving 4 other indices which can be permuted in 3 ways, so the number of terms is $5\times 3\times 1 = 15$ and so on.