SPECIAL RELATIVITY WITH THE K-CALCULUS

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Reference: d'Inverno, Ray, *Introducing Einstein's Relativity* (1992), Oxford Uni Press. - Sections 2.7, 2.8 and Problems 2.2, 2.3.

The simplest approach to special relativity that I've found is that presented in d'Inverno's book, so I'll run through the argument here and in the next few posts.

d'Inverno uses the rather cryptic title of 'The k-calculus' to introduce special relativity, which makes the reader (well, me, at any rate) think we're in for some high-powered mathematics. In fact, nothing could be further from the truth.

We start, as usual, with the two postulates of relativity:

(1) All inertial observers are equivalent.
(2) The velocity of light is the same in all inertial systems.

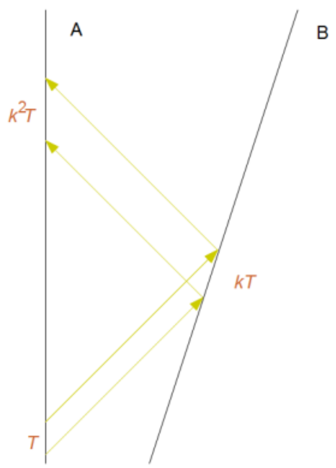
Postulate 1 actually conceals a rather subtle point. Newtonian physics appears to be based on the same principle, since Newton also assumes that the laws of physics must appear the same in all inertial systems (that is, systems moving at constant velocity relative to each other). However, Newton's implicit assumption is that all we need is that if two observers in two different inertial frames perform the same experiments, they should get the same results. Newton does not consider the role of the experimenter in these experiments; that is, that it is necessary to *observe* the experiment in order to interpret its results. This may seem a trivial point, but it is vital, since observation requires receiving signals, usually in the form of light, from the experimental apparatus. Thus the behaviour of light is of crucial importance in interpreting the experiments.

For most experiments, of course, this doesn't make much difference, since we're dealing with speeds much less than the speed of light, so assuming that light has an infinite speed is a good enough approximation. However, as we approach the speed of light, it does in fact make an enormous difference. Newton assumed implicitly that the speed of light would depend on the speed of its source (so that light from a source moving towards you would be faster than light from a stationary source), which contradicts postulate 2 above.

It's usual in relativity to take the speed of light to be 1 ($c = 1$), which effectively converts the units of distance into the units of time. Thus, one light-second (the distance light moves in a second) is written simply as 1 second, and so on.

We'll start by considering motion in one spatial dimension only. For such a system, we can represent the state of affairs using a planar space-time diagram, with the x (horizontal) axis being the space dimension and the t (vertical) axis being time. On such a diagram, an observer A who is at rest relative to us would have a constant position, and move only in time. Thus A's world line is vertical, parallel to the t axis.

A second observer B, moving at a constant speed v to the right (relative to A), has a world line at an angle to that of A, as shown in the figure:

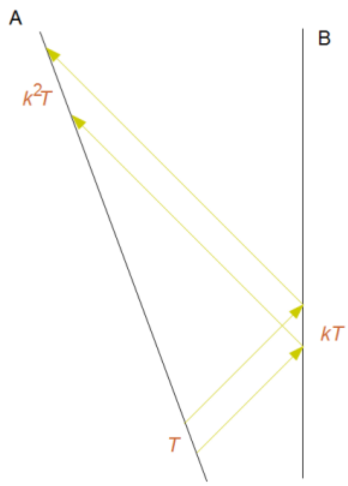


Now suppose A fires off two light pulses, separated by a time T as measured by A. Since $c = 1$, a beam of light's world line always makes an angle of $\pm\pi/4$ with the x axis (positive if the light is moving to the right, negative if to the left). Thus the two pulses emitted by A are as shown by the two yellow lines moving towards the upper right.

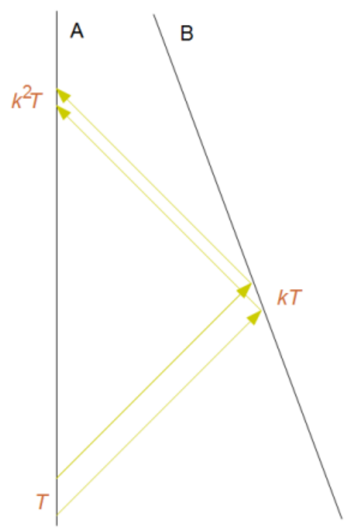
Since B is moving at a constant speed, we make the assumption that the relation between the space and time coordinates of A and B is linear. In that case, we can say that the time interval between the two pulses when they arrive at B is kT (as measured by B), where k is a constant. (This is the 'k' in the k-calculus; I told you it wasn't all that complicated.)

By the same argument, if B reflects these two light pulses back towards A, they will follow the paths shown by the two yellow arrows heading towards the upper left. Because A is moving at a speed of $-v$ relative to B, the same factor should relate the time interval between the reflections at B and their arrival at A, so in terms of the original time interval T, the time interval between the two reflected pulses will be k^2T as measured by A.

We can draw the above diagram from B's point of view:



Finally, we can look at the diagram for the case where B is moving towards A:



Note that in this case, the time intervals get smaller with each interception of the light signals, so it looks like $k < 1$ here.

We can find k in terms of v by making another simple argument. Since we're using light to make our observations, it makes sense to define the distance of an event from an observer as half the time it takes a light signal to make the round trip from the observer to the event and back again. (There wouldn't be much point in defining the distance as the time taken to go from the observer to the event, since the observer has to be able to *see* something to make the measurement, so we need the light signal to get back.) That is, if A sends off a light signal at time t_1 and receives the reflection back at time t_2, then the distance (remember we're using time units for distance) is

$$d = \frac{1}{2}(t_2 - t_1) \tag{1}$$

If we fix the origin of the coordinate system at the observer, then this is also the x coordinate:

$$x = \frac{1}{2}(t_2 - t_1) \tag{2}$$

We can also fix the time of the event to be the average of the emission and reception times, so we get

$$t = \frac{1}{2}(t_1 + t_2) \tag{3}$$

Now suppose that A and B are at the same location at $t = 0$ and synchronize their clocks at that point. Then, after a time T, A sends out a light pulse towards B. As measured by B, this pulse arrives at time kT. (We can think of this as a special case of the two-pulse experiment above, with the first pulse sent at $t = 0$ when both observers are at the same place.) If B reflects this pulse back towards A, it will arrive at A at time k^2T as measured by A. Thus A is able to work out the coordinates of the event where B reflected the pulse, using the above formulas. He gets

$$\begin{aligned} x &= \frac{1}{2}\left(k^2T - T\right) && (4)\\ &= \frac{1}{2}T\left(k^2 - 1\right) && (5)\\ t &= \frac{1}{2}T\left(k^2 + 1\right) && (6) \end{aligned}$$

Since B was at $x = 0$ when $t = 0$ and we now have x and t coordinates for B at a later time, we can work out B's speed:

$$\begin{aligned} v &= \frac{x}{t} && (7)\\ &= \frac{k^2 - 1}{k^2 + 1} && (8) \end{aligned}$$

We can solve this for k, to get

$$\left(k^{2}+1\right)v = k^{2}-1 \tag{9}$$

$$k^{2} = \frac{1+v}{1-v} \tag{10}$$

$$k = \left(\frac{1+v}{1-v}\right)^{1/2} \tag{11}$$

If $0<v<1$ (B is moving to the right away from A), then $k>1$, while if $-1<v<0$ (B moving to the left towards A), then $k<1$ as we saw in the diagrams above. In fact, it's fairly obvious that if we replace v by $-v$, $k\rightarrow1/k$. In terms of 'red shift' and 'blue shift' Doppler effects, $k>1$ is a red shift and $k<1$ a blue shift.

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