

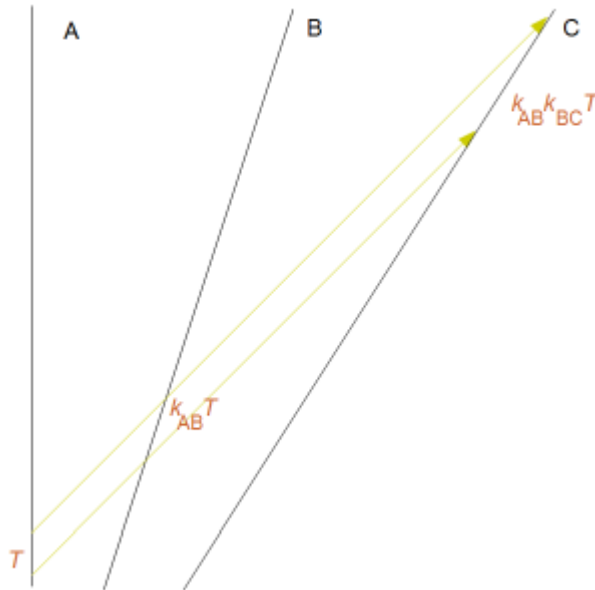
COMPOSITION OF VELOCITIES IN RELATIVITY

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Reference: d'Inverno, Ray, *Introducing Einstein's Relativity* (1992), Oxford Uni Press. - Section 2.9 and Problems 2.4, 2.5.

The composition of two velocities in special relativity has a particularly simple derivation using the k-calculus. Suppose we have 3 observers as shown in the diagram:



Observer A is at rest relative to us, while B moves to the right with velocity v_{AB} and C also moves to the right with velocity v_{AC} , with both velocities measured relative to A . Now suppose that A emits two light beams separated by a time interval T . From our k-calculus results, we know that B will receive these beams separated by a time $k_{AB}T$. If B then sends these two beams on their way to C , C will receive them at a time interval $k_{BC}(k_{AB}T)$. Thus the overall k-factor from A to C is

$$k_{AC} = k_{AB}k_{BC} \tag{1}$$

We already worked out k in terms of v , so we have

$$k_{AC} = \left(\frac{1+v_{AC}}{1-v_{AC}} \right)^{1/2} \quad (2)$$

$$= \left(\frac{1+v_{AB}}{1-v_{AB}} \right)^{1/2} \left(\frac{1+v_{BC}}{1-v_{BC}} \right)^{1/2} \quad (3)$$

Squaring this equation we get

$$\frac{1+v_{AC}}{1-v_{AC}} = \frac{1+v_{AB}}{1-v_{AB}} \frac{1+v_{BC}}{1-v_{BC}} \quad (4)$$

$$v_{AC} \left(1 + \frac{1+v_{AB}}{1-v_{AB}} \frac{1+v_{BC}}{1-v_{BC}} \right) = \frac{1+v_{AB}}{1-v_{AB}} \frac{1+v_{BC}}{1-v_{BC}} - 1 \quad (5)$$

$$v_{AC} = \frac{(1+v_{AB})(1+v_{BC}) - (1-v_{AB})(1-v_{BC})}{(1+v_{AB})(1+v_{BC}) + (1-v_{AB})(1-v_{BC})} \quad (6)$$

$$= \frac{v_{AB} + v_{BC}}{1 + v_{AB}v_{BC}} \quad (7)$$

The composition of two velocities (in the same direction) is therefore less than just the arithmetic sum. In fact, if we start with two velocities, both less than 1 (that is, less than the speed of light), then their sum is also less than 1. We can show this with a little calculus.

Consider the function, defined for $0 < x < 1$ and $0 < y < 1$:

$$f(x, y) = \frac{x+y}{1+xy} \quad (8)$$

Taking its two partial derivatives we find

$$\frac{\partial f}{\partial x} = -\frac{y(x+y)}{(1+xy)^2} + \frac{1}{x+y} \quad (9)$$

$$\frac{\partial f}{\partial y} = -\frac{x(x+y)}{(1+xy)^2} + \frac{1}{x+y} \quad (10)$$

Setting each of these to zero, we get the two conditions

$$y^2 = 1 \quad (11)$$

$$x^2 = 1 \quad (12)$$

Thus there are no maxima, minima, or saddle points anywhere inside the region, and the extreme values of the function must lie on the boundary. The boundaries are

$$f(x, 1) = \frac{1+x}{1+x} \quad (13)$$

$$= 1 \quad (14)$$

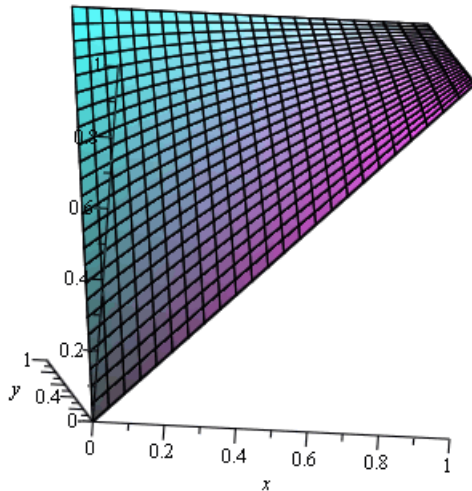
$$f(1, y) = \frac{1+y}{1+y} \quad (15)$$

$$= 1 \quad (16)$$

$$f(x, 0) = x \quad (17)$$

$$f(0, y) = y \quad (18)$$

Thus the maximum of the function occurs at the point $(x, y) = (1, 1)$ and has the value 1. A plot of the function looks like this:



The nearest corner is the origin, with the point $(1, 1)$ lying furthest away. For velocities $v \ll 1$, the formula reduces to the Newtonian formula. We can approximate the formula above using a Taylor series:

$$\frac{x+y}{1+xy} \simeq (x+y)(1-xy+\dots) \quad (19)$$

If we save only up to first-order terms, we get

$$\frac{x+y}{1+xy} \simeq x+y \quad (20)$$

or, in terms of velocities

$$v_{AC} \simeq v_{AB} + v_{BC} \quad (21)$$

For negative velocities, we can look at the region $-1 < x < 1$ and $-1 < y < 1$. The other two boundaries are

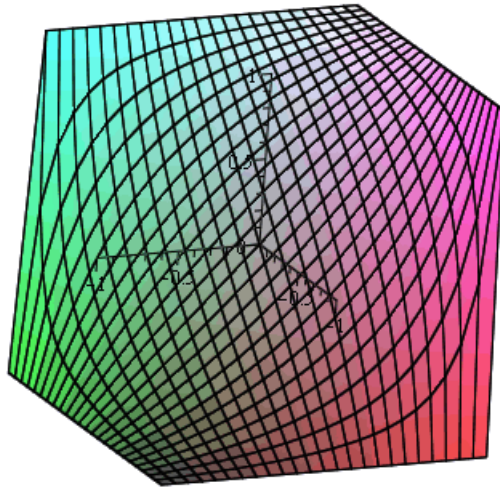
$$f(x, -1) = \frac{x-1}{1-x} \quad (22)$$

$$= -1 \quad (23)$$

$$f(-1, y) = \frac{y-1}{1-y} \quad (24)$$

$$= -1 \quad (25)$$

Thus along these boundaries, the extreme value of the function is -1 . The function is actually discontinuous at the two points $(-1, 1)$ and $(1, -1)$, where the value tends to ± 1 depending on how you approach the point. A plot looks like this:



The viewpoint is roughly the same as in the previous plot, with the nearest corner being $(-1, -1)$ and the farthest corner at the top being $(1, 1)$.

Another plot rotated about 90° to the left shows the shape a bit more clearly:

