COMPOSITION OF VELOCITIES IN RELATIVITY

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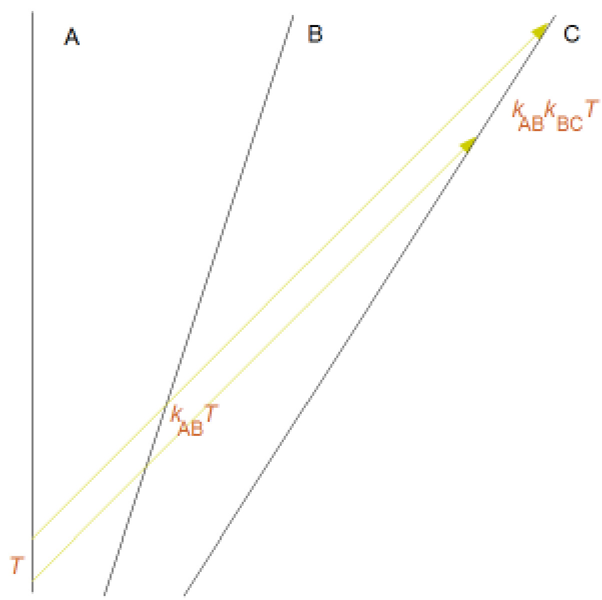
Reference: d'Inverno, Ray, *Introducing Einstein's Relativity* (1992), Oxford Uni Press. - Section 2.9 and Problems 2.4, 2.5.

The composition of two velocities in special relativity has a particularly simple derivation using the k-calculus. Suppose we have 3 observers as shown in the diagram:



Observer A is at rest relative to us, while B moves to the right with velocity v_{AB} and C also moves to the right with velocity v_{AC}, with both velocities measured relative to A. Now suppose that A emits two light beams separated by a time interval T. From our k-calculus results, we know that B will receive these beams separated by a time $k_{AB}T$. If B then sends these two beams on their way to C, C will receive them at a time interval $k_{BC}(k_{AB}T)$. Thus the overall k-factor from A to C is

$$k_{AC}=k_{AB}k_{BC} \tag{1}$$

We already worked out k in terms of v, so we have

$$k_{AC} = \left(\frac{1+v_{AC}}{1-v_{AC}}\right)^{1/2} \tag{2}$$

$$= \left(\frac{1+v_{AB}}{1-v_{AB}}\right)^{1/2}\left(\frac{1+v_{BC}}{1-v_{BC}}\right)^{1/2} \tag{3}$$

Squaring this equation we get

$$\frac{1+v_{AC}}{1-v_{AC}} = \frac{1+v_{AB}}{1-v_{AB}}\frac{1+v_{BC}}{1-v_{BC}} \tag{4}$$

$$v_{AC}\left(1+\frac{1+v_{AB}}{1-v_{AB}}\frac{1+v_{BC}}{1-v_{BC}}\right) = \frac{1+v_{AB}}{1-v_{AB}}\frac{1+v_{BC}}{1-v_{BC}} - 1 \tag{5}$$

$$v_{AC} = \frac{(1+v_{AB})(1+v_{BC})-(1-v_{AB})(1-v_{BC})}{(1+v_{AB})(1+v_{BC})+(1-v_{AB})(1-v_{BC})} \tag{6}$$

$$= \frac{v_{AB}+v_{BC}}{1+v_{AB}v_{BC}} \tag{7}$$

The composition of two velocities (in the same direction) is therefore less than just the arithmetic sum. In fact, if we start with two velocities, both less than 1 (that is, less than the speed of light), then their sum is also less than 1. We can show this with a little calculus.

Consider the function, defined for $0 < x < 1$ and $0 < y < 1$:

$$f(x,y) = \frac{x+y}{1+xy} \tag{8}$$

Taking its two partial derivatives we find

$$\frac{\partial f}{\partial x} = -\frac{y(x+y)}{(1+xy)^2} + \frac{1}{x+y} \tag{9}$$

$$\frac{\partial f}{\partial y} = -\frac{x(x+y)}{(1+xy)^2} + \frac{1}{x+y} \tag{10}$$

Setting each of these to zero, we get the two conditions

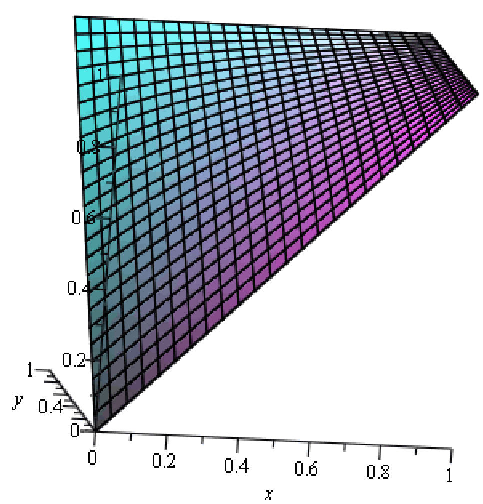
$$y^2 = 1 \tag{11}$$

$$x^2 = 1 \tag{12}$$

Thus there are no maxima, minima, or saddle points anywhere inside the region, and the extreme values of the function must lie on the boundary. The boundaries are

$$f(x,1) = \frac{1+x}{1+x} \tag{13}$$
$$= 1 \tag{14}$$
$$f(1,y) = \frac{1+y}{1+y} \tag{15}$$
$$= 1 \tag{16}$$
$$f(x,0) = x \tag{17}$$
$$f(0,y) = y \tag{18}$$

Thus the maximum of the function occurs at the point $(x,y) = (1,1)$ and has the value 1. A plot of the function looks like this:



The nearest corner is the origin, with the point $(1,1)$ lying furthest away.

For velocities $v \ll 1$, the formula reduces to the Newtonian formula. We can approximate the formula above using a Taylor series:

$$\frac{x+y}{1+xy} \simeq (x+y)(1-xy+\ldots) \tag{19}$$

If we save only up to first-order terms, we get

$$\frac{x+y}{1+xy} \simeq x+y \tag{20}$$

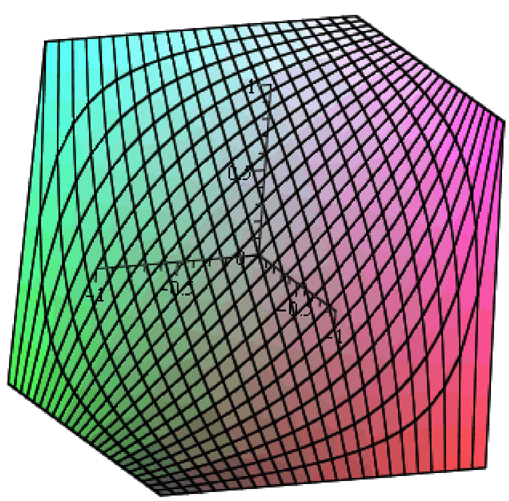
or, in terms of velocities

$$v_{AC} \simeq v_{AB} + v_{BC} \tag{21}$$

For negative velocities, we can look at the region $-1 < x < 1$ and $-1 < y < 1$. The other two boundaries are

$$\begin{aligned}
f(x,-1) &= \frac{x-1}{1-x} && (22)\\
&= -1 && (23)\\
f(-1,y) &= \frac{y-1}{1-y} && (24)\\
&= -1 && (25)
\end{aligned}$$

Thus along these boundaries, the extreme value of the function is -1. The function is actually discontinuous at the two points $(-1,1)$ and $(1,-1)$, where the value tends to ± 1 depending on how you approach the point. A plot looks like this:



The viewpoint is roughly the same as in the previous plot, with the nearest corner being $(-1,-1)$ and the farthest corner at the top being $(1,1)$.

Another plot rotated about 90° to the left shows the shape a bit more clearly: