CONTRAVARIANT TENSORS

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Reference: d'Inverno, Ray, *Introducing Einstein's Relativity* (1992), Oxford Uni Press. - Section 5.5 and Problems 5.5, 5.6.

Suppose we have a function defined in an n-dimensional manifold:

$$f=f(x) \tag{1}$$

where x represents all n coordinates in some coordinate system.

Now suppose we define some curve within the manifold, and ask how the function varies as we move along this curve. Since the curve is one-dimensional, it can be described using a single parameter u. Then the rate of change of f along u can be found using the chain rule:

$$\frac{df}{du}=\frac{\partial f}{\partial x^i}\frac{dx^i}{du} \tag{2}$$

where we're using the summation convention. We've used the total derivative notation for the derivatives with respect to u, since in both these cases, we're considering a derivative along the particular path that is given by the single parameter u, so there is only one independent variable in those cases. The derivative with respect to x^i must be partial, since f depends on (in general) all the x^is.

In particular, we can consider a change of coordinates, and write each of the new, primed coordinates as a function of the old unprimed coordinates, like so:

$$x'^a=x'^a(x) \tag{3}$$

Using the chain rule formula, we find that

$$\frac{dx'^a}{du}=\frac{\partial x'^a}{\partial x^i}\frac{dx^i}{du} \tag{4}$$

Dropping the du off both sides (the way physicists do, and mathematicians hate), we get an expression for the transformation of the differentials between two coordinate systems:

$$dx'^a = \frac{\partial x'^a}{\partial x^i} dx^i \tag{5}$$

Any quantity that transforms in this way is called a *contravariant tensor of rank 1*, or, for short, a *contravariant vector*.

The tangent vector $\mathbf{t}(u)$ (from elementary calculus) to a parametric curve given in vector form is the derivative of each component of the curve's vector with respect to u, and has components in a given coordinate system:

$$t^a(u) = \frac{dx^i}{du} \tag{6}$$

Thus the tangent vector is a contravariant vector.

As an example, suppose we have a circle of radius a centred at the origin. We can write this in rectangular coordinates as a curve using the angle θ as the parameter in the usual way:

$$x = a\cos\theta \tag{7}$$
$$y = a\sin\theta \tag{8}$$

In this system, the tangent vector is then

$$t(\theta) = \frac{d}{d\theta}\begin{pmatrix} x & y \end{pmatrix} = \begin{pmatrix} -a\sin\theta & a\cos\theta \end{pmatrix} = \begin{pmatrix} -y & x \end{pmatrix} \tag{9}$$

This makes sense, since the tangent to a circle is perpendicular to the radius vector, and the radius vector has coordinates $\begin{pmatrix} x & y \end{pmatrix}$, so the dot product of radius and tangent gives zero as required.

In polar coordinates (which we'll take as the primed system), the tangent is exceptionally easy to find, since the parametric equations are

$$r = a \tag{10}$$
$$\theta = \theta \tag{11}$$

so the tangent is

$$t'(\theta) = \frac{d}{d\theta}\begin{pmatrix} r & \theta \end{pmatrix} = \begin{pmatrix} 0 & 1 \end{pmatrix} \tag{12}$$

However, we can use the tensor transformation equation above to check this. We've already done the calculation of the transformation matrix when we considered Jacobians earlier, so we get

$$\begin{aligned} r &= \sqrt{x^2+y^2} && (13) \\ \theta &= \tan^{-1}\frac{y}{x} && (14) \end{aligned}$$

so

$$\frac{\partial x'^a}{\partial x^i} = \left[\begin{array}{cc} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y/x^2}{1+(y/x)^2} & \frac{1/x}{1+(y/x)^2} \end{array}\right] = \left[\begin{array}{cc} \cos\theta & \sin\theta \\ -\frac{\sin\theta}{a} & \frac{\cos\theta}{a} \end{array}\right] \quad (15)$$

If we multiply the terms out, we get

$$t'(\theta) = \left[\begin{array}{cc} \cos\theta & \sin\theta \\ -\frac{\sin\theta}{a} & \frac{\cos\theta}{a} \end{array}\right] \left[\begin{array}{c} -a\sin\theta \\ a\cos\theta \end{array}\right] = \left[\begin{array}{c} 0 \\ 1 \end{array}\right] \quad (16)$$

Contravariant tensors of higher rank can be defined by a simple extension of the above formula. A rank-2 tensor transforms according to:

$$X'^{ab} = \frac{\partial x'^a}{\partial x^i}\frac{\partial x'^b}{\partial x^j}X^{ij} \quad (17)$$

Higher ranks are defined by just adding in more transformation matrixes, as you'd expect.

We can define the product of two contravariant vectors as

$$T^{ab} = X^a X^b \quad (18)$$

Note that this is *not* a dot or cross product, since there are no repeated indices on the right, so no implied summation. This equation is shorthand for an $n\times n$ matrix with components given by the equation tiself. For example, the top row of the matrix has elements X^1X^1, $X^1X^2, \ldots X^1X^n$, with similar results for the other rows.

This quantity transforms by applying the formula above for vectors:

$$\begin{aligned} T'^{ab} &= X'^a X'^b && (19) \\ &= \left(\frac{\partial x'^a}{\partial x^i}X^i\right)\left(\frac{\partial x'^b}{\partial x^j}X^j\right) && (20) \\ &= \frac{\partial x'^a}{\partial x^i}\frac{\partial x'^b}{\partial x^j}X^iX^j && (21) \end{aligned}$$

Thus the product of two vectors has the required transformation property. By the same argument, the product of m contravariant vectors is a rank-m contravariant tensor.

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