COVARIANT AND MIXED TENSORS

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Reference: d'Inverno, Ray, *Introducing Einstein's Relativity* (1992), Oxford Uni Press. - Section 5.6 and Problem 5.7.

We've seen that objects such as the tangent vector to a curve are contravariant tensors, in that they transform under a change of coordinates according to the rule (for a rank-2 tensor, for example):

$$X'^{ab}=\frac{\partial x'^a}{\partial x^i}\frac{\partial x'^b}{\partial x^j}X^{ij} \tag{1}$$

Note that the indices on the tensor are superscripts, and in the transformation, the original coordinate system's components are those with which the derivative is taken with respect to. That is, the *new* (primed) coordinates are taken to be functions of the *old* (unprimed) coordinates.

Suppose we turn the tables and express the unprimed coordinates as functions of the primed ones, like so:

$$x^a=x^a\left(x'\right) \tag{2}$$

where as usual this notation indicates a set of n equations, one for each of the x^a, and the argument of the function indicates that it is a function of all the primed coordinates. This is the inverse of the original transformation from unprimed to primed coordinates.

Now suppose we have a function defined in terms of the unprimed coordinates:

$$g=g\left(x\right) \tag{3}$$

We can write this as a function of the primed coordinates using the transformation equations above:

$$g=g\left(x\left(x'\right)\right) \tag{4}$$

When we derived the condition for a contravariant tensor, we considered a one-dimensional curve defined within the manifold by using a single parameter u, and then we asked how the function changed as we moved along this curve. This time, we ask simply for the derivative of a function with respect to each of the primed coordinates. Using the chain rule, we get

$$\frac{\partial g}{\partial x'^a} = \frac{\partial g}{\partial x^i}\frac{\partial x^i}{\partial x'^a} \tag{5}$$

That is, the quantity $\partial g/\partial x^i$ transforms by multiplying it with the term $\partial x^i/\partial x'^a$ and summing over i. The quantities $\partial x^i/\partial x'^a$ are entries in the Jacobian determinant for the inverse transformation. Furthermore the index i in $\partial g/\partial x^i$ is now on the bottom (it is a superscript index, but it's in the denominator, so it counts as a lower index). We can write any object that transforms in this way using the notation T_a, so that

$$T'_a = \frac{\partial x^i}{\partial x'^a} T_i \tag{6}$$

This is called a *covariant vector,* or *covariant tensor of rank 1*. Higher rank tensors can be defined in the usual way, by multiplying by further derivative factors. Thus a rank 2 covariant tensor transforms as

$$T'_{ab} = \frac{\partial x^i}{\partial x'^a}\frac{\partial x^j}{\partial x'^b} T_{ij} \tag{7}$$

and so on.

We can also define mixed tensors (tensors that contain both contravariant and covariant indexes) in a relatively obvious way. For example, a tensor with contravariant rank 2 and covariant rank 1, written as a (2,1) tensor, is defined by

$$T'^{ab}{}_{c} = \frac{\partial x'^a}{\partial x^i}\frac{\partial x'^b}{\partial x^j}\frac{\partial x^k}{\partial x'^c} T^{ij}{}_{k} \tag{8}$$

Note the position of the primed and unprimed coordinates in each case. The summation convention applies only to repeated indexes where one index in each pair is upper and the other is lower.

Incidentally, the relative positioning of the indexes in the tensor symbol seems to be largely a matter of taste. That is, some (well, most, actually) books write the indexes so that there is a space in the bottom to avoid overlap with the top indexes (as in $T^{ab}{}_{c}$) while other books leave out the space, so we get T_c^{ab}. Since a tensor is defined in terms of its transformation equation, and the order in which we write the derivatives doesn't matter, it shouldn't matter much whether we insert a space in the notation or not, since the order of the indexes doesn't matter. What *is* important is which indexes are upper (contravariant) and which are lower (covariant).

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