KRONECKER DELTA AS A TENSOR

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Reference: d'Inverno, Ray, *Introducing Einstein's Relativity* (1992), Oxford Uni Press. - Section 5.6 and Problem 5.8.

The Kronecker delta δ_b^a is actually a tensor, as it transforms according to the rules for such a tensor, as we'll see. Recall that the rule for transformation of a contravariant vector(rank-1 tensor) is

$$T'^a = \frac{\partial x'^a}{\partial x^i} T^i \tag{1}$$

and for a covariant vector:

$$T'_a = \frac{\partial x^i}{\partial x'^a} T_i \tag{2}$$

For a mixed tensor, we multiply by the right combination of derivatives to effect a coordinate transformation. If δ_b^a really is a tensor, then it should transform properly. The numerical value of δ_b^a is

$$\delta_b^a = \begin{cases} 1 & a = b \\ 0 & a \neq b \end{cases} \tag{3}$$

Let's have a look at the transformation expression

$$\begin{aligned} \delta_b'^a &= \frac{\partial x'^a}{\partial x^i} \frac{\partial x^j}{\partial x'^b} \delta_j^i & (4) \\ &= \frac{\partial x'^a}{\partial x^i} \frac{\partial x^i}{\partial x'^b} & (5) \\ &= \frac{\partial x'^a}{\partial x'^b} & (6) \\ &= \delta_b^a & (7) \end{aligned}$$

The third line follows from the second by recognizing that the second line is the chain rule expression for the derivative in the third line. The last line follows from the fact that all the x' coordinates are independent of each other, so the derivative of any of them with respect to any other coordinate is zero, while the derivative of a coordinate with respect to itself is 1.

Thus not only does δ_b^a transform as a tensor, but it also has the same numerical value in all coordinate systems.

Note that this derivation does *not* work if we try to make delta to be either a pure contravariant or pure covariant tensor. If we start by defining (in some unprimed coordinate system):

$$\delta^{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \tag{8}$$

then applying the tensor transformation rule, we get

$$\delta'^{ab} = \frac{\partial x'^a}{\partial x^i}\frac{\partial x'^b}{\partial x^j}\delta^{ij} \tag{9}$$

$$= \sum_i \frac{\partial x'^a}{\partial x^i}\frac{\partial x'^b}{\partial x^i} \tag{10}$$

Although the index i is repeated, both occurrences are as a lower index so we can't use the summation convention, hence the explicit summation sign. However, this expression does not represent the chain rule derivative of anything in particular, and can't be simplified further. The expression depends on the coordinate systems, so this delta is not a numerical invariant, and doesn't transform like a tensor.

However, since δ_b^a is a tensor, we can raise or lower its indices using the metric tensor in the usual way. That is, we *can* get a version of δ with both indices raised or lowered, as follows:

$$\delta^{ab} = g^{cb}\delta_c^a = g^{ab} \tag{11}$$

$$\delta_{ab} = g_{ac}\delta_b^c = g_{ab} \tag{12}$$

In this sense, δ^{ab} and δ_{ab} are the upper and lower versions of the metric tensor. However, they can't really be considered versions of the Kronecker delta any more, as they don't necessarily satisfy 8. In other words, the only version of δ that is both a Kronecker delta *and* a tensor is the version with one upper and one lower index: δ_b^a.

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