TENSOR ARITHMETIC

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Reference: d'Inverno, Ray, *Introducing Einstein's Relativity* (1992), Oxford Uni Press. - Section 5.8 and Problems 5.10 - 5.14.

Since tensors are generalizations of vectors, it's not surprising that many of the arithmetic properties of vectors that you may be familiar with also apply to tensors.

We've seen that the tangent to a curve is a contravariant tensor (vector, actually), since it transforms according to

$$\frac{dx'^a}{du}=\frac{\partial x'^a}{\partial x^i}\frac{dx^i}{du} \tag{1}$$

Also, the first derivative of a function is a covariant tensor, as it transforms according to

$$\frac{\partial g}{\partial x'^a}=\frac{\partial x^i}{\partial x'^a}\frac{\partial g}{\partial x^i} \tag{2}$$

Because these transformation rules are linear, the linear arithmetic operations of addition, subtraction and multiplication by a scalar, when applied to tensors, yield other tensors.

For example, if we have two tensors Y^a_{bc} and Z^a_{bc}, their sum is also a tensor of the same type, which we can call X^a_{bc}:

$$\begin{aligned}
X'^a_{bc} &= Y'^a_{bc}+Z'^a_{bc} && (3)\\
&= \frac{\partial x'^a}{\partial x^i}\frac{\partial x^j}{\partial x'^b}\frac{\partial x^k}{\partial x'^c}Y^i_{\;jk}+\frac{\partial x'^a}{\partial x^i}\frac{\partial x^j}{\partial x'^b}\frac{\partial x^k}{\partial x'^c}Z^i_{\;jk} && (4)\\
&= \frac{\partial x'^a}{\partial x^i}\frac{\partial x^j}{\partial x'^b}\frac{\partial x^k}{\partial x'^c}\left(Y^i_{\;jk}+Z^i_{\;jk}\right) && (5)\\
&= \frac{\partial x'^a}{\partial x^i}\frac{\partial x^j}{\partial x'^b}\frac{\partial x^k}{\partial x'^c}X^i_{\;jk} && (6)
\end{aligned}$$

Obviously the same argument works for the difference of two tensors.

Note that it makes sense to define addition and subtraction only between tensors of the same type. In elementary linear algebra, for example, it makes no sense to talk about the sum or difference of a scalar and a vector, since the two objects are fundamentally different and cannot be combined in this way.

A second rank tensor such as X^{ab} is symmetric if $X^{ab} = X^{ba}$ for all pairs of indexes. A tensor is anti-symmetric if $X^{ab} = -X^{ba}$. Symmetry and anti-symmetry are examples of *tensorial properties*, which are properties that, if true in one coordinate system, are true in all coordinate systems. For symmetry, we have

$$\begin{aligned} X'^{ab} &= \frac{\partial x'^a}{\partial x^i}\frac{\partial x'^b}{\partial x^j}X^{ij} && (7)\\ &= \frac{\partial x'^a}{\partial x^i}\frac{\partial x'^b}{\partial x^j}X^{ji} && (8)\\ &= X'^{ba} && (9)\end{aligned}$$

The argument for anti-symmetry is the same, except a minus sign is introduced in the second line.

If X^{ab} is anti-symmetric and Y_{ab} is symmetric, then their product (with implied summation) is

$$\begin{aligned} X^{ab}Y_{ab} &= -X^{ba}Y_{ba} && (10)\\ &= -X^{ab}Y_{ab} && (11)\end{aligned}$$

The first line follows from the anti-symmetry of X^{ab}. The second line follows from the fact that a and b are dummy indexes so it doesn't matter what we call them, so we are justified in just swapping them around. Thus the equation has the form $X^{ab}Y_{ab} = -X^{ab}Y_{ab}$, which means that $X^{ab}Y_{ab} = 0$. This is because the terms in the sum cancel each other in pairs. (Note that for an anti-symmetric tensor, $X^{aa} = -X^{aa} = 0$.)

A tensor can be split into a sum of symmetric and anti-symmetric parts. For a rank 2 tensor, clearly the following tensor is symmetric:

$$X^{(ab)} \equiv \frac{1}{2}\left(X^{ab} + X^{ba}\right) \tag{12}$$

and the following tensor is anti-symmetric:

$$X^{[ab]} \equiv \frac{1}{2}\left(X^{ab} - X^{ba}\right) \tag{13}$$

The special bracket notation is used to denote symmetric and anti-symmetric tensors.

The sum gives us back the original tensor:

$$X^{ab} = X^{(ab)} + X^{[ab]} \tag{14}$$

The notion of symmetry can be extended to tensors of arbitrary rank:

$$X_{(a_1 a_2 \ldots a_r)} = \frac{1}{r!}\sum(\text{all permutations of the indexes}) \tag{15}$$

$$X_{[a_1 a_2 \ldots a_r]} = \frac{1}{r!}\left[\sum(\text{all even permutations}) - \sum(\text{all odd permutations})\right] \tag{16}$$

An even permutation is one where an even number of swaps is needed to get from the original order to a given order, and an odd permutation requires an odd number of swaps. The even permutations of the starting index order abc are cab and bca, leaving bac, cba and acb as the odd permutations. A symmetric tensor is one where all components with a particular set of indexes, permuted in any order, are equal.

A tensor can be *contracted* by setting a pair of one upper and one lower index equal, giving a summation. For example, given a rank 3 tensor $X^{a}_{\ bc}$ we can form the contraction $Y_c = X^{a}_{\ ac}$. The result of a contraction is another tensor, as we can verify by finding its transformation:

$$\begin{aligned}
Y'_c &= X'^{a}_{\ ac} && (17)\\
&= \frac{\partial x'^a}{\partial x^i}\frac{\partial x^j}{\partial x'^a}\frac{\partial x^k}{\partial x'^c}X^{i}_{\ jk} && (18)\\
&= \frac{\partial x^j}{\partial x^i}\frac{\partial x^k}{\partial x'^c}X^{i}_{\ jk} && (19)\\
&= \delta^j_i\frac{\partial x^k}{\partial x'^c}X^{i}_{\ jk} && (20)\\
&= \frac{\partial x^k}{\partial x'^c}X^{i}_{\ ik} && (21)\\
&= \frac{\partial x^k}{\partial x'^c}Y_k && (22)
\end{aligned}$$

Thus the contracted tensor transforms as a covariant vector.

We've seen that the Kronecker delta is a tensor if its indexes are written as δ^a_b. If we contract this tensor, we get (in an n-dimensional manifold)

$$\delta^a_a = \sum_{a=1}^{n} 1 = n \tag{23}$$

How about $\delta^a_b\delta^b_a$? This is easiest to see by writing out the sums.

$$\delta^a_b \delta^b_a = \sum_{a=1}^{n} \sum_{b=1}^{n} \delta^a_b \delta^b_a \tag{24}$$

$$= \sum_{a=1}^{n} \delta^a_a \delta^a_a \tag{25}$$

$$= \sum_{a=1}^{n} 1 \times 1 \tag{26}$$

$$= n \tag{27}$$

The second line follows since $\delta^a_b \neq 0$ only when $b = a$.

PINGBACKS

Pingback: Congruence of curves