LIE BRACKETS (COMMUTATORS)

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Reference: d'Inverno, Ray, *Introducing Einstein's Relativity* (1992), Oxford Uni Press. - Section 5.9 and Problems 5.15, 5.16 (v).

When we began looking at quantum mechanics, we encountered the commutator of two operators, defined as

$$[A,B] \equiv AB - BA \tag{1}$$

In quantum mechanics, some operators (the most famous being the position and momentum operators) do not commute, and in fact, the generalized uncertainty principle says that only operators that commute can be measured simultaneously with arbitrary precision.

In tensor analysis, we've seen that the tangent vector field to a manifold can be written as the operator

$$X = X^a \partial_a \tag{2}$$

Since this operator involves derivatives, we might expect that the commutator of two such operators would be non-zero (since that's what happend with the position and momentum operators in quantum mechanics). The commutator of two vector fields is also known as a *Lie bracket*, (where 'Lie' is pronounced 'lee') but is defined in the same way as in quantum mechanics.

The commutator of two vector fields is again a vector field, as can be verified by direct calculation. As always with operators involving derivatives, we need a dummy function f on which to operate, so we get

$$[X,Y]f = X^a \partial_a \left(Y^b \partial_b f\right) - Y^a \partial_a \left(X^b \partial_b f\right) \tag{3}$$

$$= X^a \left(\partial_a Y^b\right)(\partial_b f) + X^a Y^b \partial^2_{ab} f - Y^a \left(\partial_a X^b\right)(\partial_b f) - Y^a X^b \partial^2_{ab} f \tag{4}$$

$$= X^a \left(\partial_a Y^b\right)(\partial_b f) - Y^a \left(\partial_a X^b\right)(\partial_b f) \tag{5}$$

Removing the dummy function, we get

$$[X,Y] = X^a \left(\partial_a Y^b\right) \partial_b - Y^a \left(\partial_a X^b\right) \partial_b \tag{6}$$

which is a vector field with components

$$[X,Y]^b = X^a \partial_a Y^b - Y^a \partial_a X^b \tag{7}$$

It's obvious from the definition that

$$[X,X] = 0 \tag{8}$$
$$[X,Y] = -[Y,X] \tag{9}$$

There is a third identity known as *Jacobi's identity* that is less obvious:

$$[X,[Y,Z]] + [Z,[X,Y]] + [Y,[Z,X]] = 0 \tag{10}$$

This is true for commutators in general, and not just for vector fields as defined here. It can be proved by writing out the terms.

$$\begin{aligned} [X,[Y,Z]] &= XYZ - XZY - YZX + ZYX \\ [Z,[X,Y]] &= ZXY - ZYX - XYZ + YXZ \\ [Y,[Z,X]] &= YZX - YXZ - ZXY + XZY \end{aligned}$$

Adding up the right hand side, we see that the terms cancel in pairs.

As an example of Lie brackets, we can look at the operators X, Y and Z that we used in the post on tangent space. In rectangular coordinates, these operators are

$$X = \partial_x \tag{11}$$
$$Y = \partial_y \tag{12}$$
$$Z = -y\partial_x + x\partial_y \tag{13}$$

To work out the commutators, we can use equation 7 above. For that, we need the components of the vectors, which are $X^a = (1,0)$, $Y^a = (0,1)$ and $Z^a = (-y,x)$.

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\begin{align}
[X,Y]^b &= X^a\partial_a Y^b - Y^a\partial_a X^b \tag{14}\\
&= (0,0) \tag{15}\\
[X,Z]^1 &= X^a\partial_a Z^1 - Z^a\partial_a X^1 \tag{16}\\
&= \partial_x(-y)+0-(0+0) \tag{17}\\
&= 0 \tag{18}\\
[X,Z]^2 &= X^a\partial_a Z^2 - Z^a\partial_a X^2 \tag{19}\\
&= \partial_x x+0-(0+0) \tag{20}\\
&= 1 \tag{21}\\
[Y,Z]^1 &= Y^a\partial_a Z^1 - Z^a\partial_a Y^1 \tag{22}\\
&= 0+\partial_y(-y)-(0+0) \tag{23}\\
&= -1 \tag{24}\\
[Y,Z]^2 &= Y^a\partial_a Z^2 - Z^a\partial_a Y^2 \tag{25}\\
&= 0+\partial_y x-(0+0) \tag{26}\\
&= 0 \tag{27}
\end{align}
$$

Thus the commutator operators are

$$
\begin{align}
[X,Y] &= 0 \tag{28}\\
[X,Z] &= \partial_y = Y \tag{29}\\
[Y,Z] &= -\partial_x = -X \tag{30}
\end{align}
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