TANGENT SPACE

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Reference: d'Inverno, Ray, *Introducing Einstein's Relativity* (1992), Oxford Uni Press. - Section 5.9 and Problems 5.16 (i - iv).

When we looked at contravariant vectors, we examined the case of the directional derivative along a curve, and showed that if we have the curve defined parametrically by

$$f = f(x^a(u)) \tag{1}$$

where u is the parameter, then the directional derivative is given by (using the chain rule):

$$\frac{df}{du} = \frac{\partial f}{\partial x^i}\frac{dx^i}{du} \tag{2}$$

In 3-d Euclidean geometry, the derivative df/du is the tangent to the curve. By drawing all possible curves through a particular point P and taking the directional derivative of each curve, we can generate a collection of tangents called the *tangent space*. In 3-d Euclidean geometry, the tangent space can be visualized as the plane tangent to a surface at P. In higher dimensions and in non-Euclidean geometries, the concept is not so easy to visualize, but the mathematics generalizes in a straightforward way.

It's important to note that in non-Euclidean geometries, the space in which the tangent space lies may be different from that of the space which it is tangent to. Examples of this must be delayed until we've looked at such non-Euclidean spaces.

We can rewrite the above equation as an operator equation:

$$\frac{df}{du} = Xf \tag{3}$$

where

$$X \equiv \frac{dx^i}{du}\frac{\partial}{\partial x^i} \tag{4}$$

As a shorthand notation, since the partial derivative turns up often, we can write

$$
\begin{aligned}
\partial_i &\equiv \frac{\partial}{\partial x^i} && (5)\\
X &= \frac{dx^i}{du}\partial_i && (6)
\end{aligned}
$$

As a further shorthand, we can define the components of the contravariant vector to be $X^a \equiv dx^a/du$, so we get

$$
X = X^a \partial_a \qquad (7)
$$

This operator is invariant under a change of coordinates, as can be shown fairly easily. Since the quantities X^a make up a contravariant vector, we know how they transform, so we get

$$
\begin{aligned}
X'^a \partial'_a &= \frac{\partial x'^a}{\partial x^i} X^i \partial'_a && (8)\\
&= \frac{\partial x'^a}{\partial x^i} X^i \frac{\partial x^j}{\partial x'^a}\partial_j && (9)\\
&= \delta_i^j X^i \partial_j && (10)\\
&= X^i \partial_i && (11)
\end{aligned}
$$

As an example, suppose we have a vector field in two dimensions given in rectangular coordinates as $X^a = (1,0)$. That is, the vector field 'points' in the x direction and has a constant magnitude over the entire plane. To transform to polar coordinates, we can use the contravariant transformation

$$
X'^a = \frac{\partial x'^a}{\partial x^i} X^i \qquad (12)
$$

For polar coordinates $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}(y/x)$ we get

$$
\begin{aligned}
X'^1 &= \frac{\partial r}{\partial x} X^1 + 0 && (13)\\
&= \frac{x}{r} && (14)\\
&= \cos\theta && (15)\\
X'^2 &= \frac{\partial \theta}{\partial x} X^1 + 0 && (16)\\
&= -\frac{\sin\theta}{r} && (17)
\end{aligned}
$$

The gradient operator can be written in the two coordinate systems as

$$\nabla f = \frac{\partial f}{\partial x}\hat{\mathbf{i}}+\frac{\partial f}{\partial y}\hat{\mathbf{j}} \tag{18}$$
$$= \frac{\partial f}{\partial r}\hat{\mathbf{r}}+\frac{\partial f}{\partial \theta}\frac{\hat{\theta}}{r} \tag{19}$$

The relation between the unit vectors is

$$\hat{\mathbf{r}} = \cos\theta\hat{\mathbf{i}}+\sin\theta\hat{\mathbf{j}} \tag{20}$$
$$\hat{\theta} = -\sin\theta\hat{\mathbf{i}}+\cos\theta\hat{\mathbf{j}} \tag{21}$$

If we take the dot product of ∇f with each unit vector, we can get relations between the derivatives. We get

$$\nabla f\cdot\hat{\mathbf{r}} = \cos\theta\partial_x f+\sin\theta\partial_y f \tag{22}$$
$$= \partial_r f \tag{23}$$
$$\nabla f\cdot\hat{\theta} = -\sin\theta\partial_x f+\cos\theta\partial_y f \tag{24}$$
$$= \frac{1}{r}\partial_\theta f \tag{25}$$
$$\nabla f\cdot\hat{\mathbf{i}} = \cos\theta\partial_r f-\frac{\sin\theta}{r}\partial_\theta f \tag{26}$$
$$= \partial_x f \tag{27}$$
$$\nabla f\cdot\hat{\mathbf{j}} = \sin\theta\partial_r f-\frac{\cos\theta}{r}\partial_\theta f \tag{28}$$
$$= \partial_y f \tag{29}$$

In rectangular coordinates, the operator X is

$$X = X^a\partial_a \tag{30}$$
$$= \partial_x \tag{31}$$

In polar coordinates, we get

$$X' = X'^a\partial'_a \tag{32}$$
$$= \cos\theta\partial_r-\frac{\sin\theta}{r}\partial_\theta \tag{33}$$
$$= \cos^2\theta\partial_x+\cos\theta\sin\theta\partial_y+\sin^2\theta\partial_x-\cos\theta\sin\theta\partial_y \tag{34}$$
$$= \partial_x \tag{35}$$

Thus the operator is indeed the same in the two coordinate systems.

For a second vector field $Y^a = (0,1)$ we can do the same analysis to find that

$$Y'^a = \left(\sin\theta, \frac{\cos\theta}{r}\right) \tag{36}$$

$$Y = \partial_y \tag{37}$$

$$= \sin\theta\partial_r + \frac{\cos\theta}{r}\partial_\theta \tag{38}$$

Finally, if $Z^a = (-y, x)$ we get

$$Z'^a = \left(-y\cos\theta + x\sin\theta, y\frac{\sin\theta}{r} + x\frac{\cos\theta}{r}\right) \tag{39}$$

$$= (0,1) \tag{40}$$

$$Z = -y\partial_x + x\partial_y \tag{41}$$

$$= \partial_\theta \tag{42}$$

PINGBACKS

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