LIE DERIVATIVE: HIGHER-RANK TENSORS

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Reference: d'Inverno, Ray, *Introducing Einstein's Relativity* (1992), Oxford Uni Press. - Section 6.2; Problem 6.1.

We looked at a conceptual derivation of the Lie derivative earlier, but now we can look at another derivation which allows us to generalize the derivative to tensors of any rank.

The idea of a Lie derivative is that we use a vector field X to provide a congruence of curves along which derivatives of a tensor are calculated. If we have a tensor field defined over a manifold, then that tensor field has a particular value at each point in the manifold. Suppose for the sake of illustration we take a mixed tensor with one contravariant and one covariant index: T_b^a. Then at two neighbouring points P and Q, this tensor has values $T_b^a(P)$ and $T_b^a(Q)$. Remember that in the case where we were finding the derivative of a vector field (a rank-one tensor field), this vector field consists of vectors that are tangent to a congruence of curves (a different congruence from the one we're using to define the direction of the derivative - see the earlier post on the Lie derivative). In the case of a higher-rank tensor field, the tensors it contains are in a sense tangents to higher-dimensional surfaces. Thus the tensors $T_b^a(P)$ and $T_b^a(Q)$ are tangents to a surface in the 'congruence of surfaces' produced by the tensor field.

Returning to our vector field X that is being used to define the direction of the derivative, we can look at the curve from its congruence that passes through P and Q (we're assuming that we've chosen the vector field in such a way that one of its congruence of curves *does* pass through these two points). Let's say that Q is a distance δu along this curve from P, that is, if P has coordinates x^a, then Q has coordinates

$$x'^a = x^a + \delta u X^a(x) \tag{1}$$

If we apply this transformation to all points on the surface from the congruence of T_b^a that passes through P, we'll generate a new surface that passes through Q. In general, this surface will *not* be a surface from the congruence of T_b^a, so if we define $T_b'^a$ to be the tensor that is tangent to this new surface, then in general, $T_b'^a(x') \neq T_b^a(x')$. That is, the tensor obtained by dragging the surface along from P to Q by means of the transformation

1 will be different from the tensor at Q which is defined as a tangent to the surface at Q which is in the congruence of T_b^a.

We can calculate these two tensors as follows. We treat the dragging operation from P to Q as a coordinate transformation, so under this transformation we have, using the rules for transforming covariant and contravariant components:

$$T_b^{\prime a} = \frac{\partial x^{\prime a}}{\partial x^c}\frac{\partial x^d}{\partial x^{\prime b}}T_d^c \tag{2}$$

We need expressions for these two partial derivatives. We can get both by taking the derivative of 1. First, we get

$$\frac{\partial x^{\prime a}}{\partial x^c} = \delta_c^a + \delta u \frac{\partial X^a}{\partial x^c} \tag{3}$$

For the other derivative, we get

$$x^d = x^{\prime d} - \delta u X^d \tag{4}$$

$$\frac{\partial x^d}{\partial x^{\prime b}} = \delta_b^d - \delta u \frac{\partial X^d}{\partial x^{\prime b}} \tag{5}$$

We can use the chain rule on the last line to get

$$\frac{\partial x^d}{\partial x^{\prime b}} = \delta_b^d - \delta u \frac{\partial X^d}{\partial x^c}\frac{\partial x^c}{\partial x^{\prime b}} \tag{6}$$

$$= \delta_b^d - \delta u \frac{\partial X^d}{\partial x^c}\left(\delta_b^c - \delta u \frac{\partial X^c}{\partial x^{\prime b}}\right) \tag{7}$$

$$= \delta_b^d - \delta u \frac{\partial X^d}{\partial x^b} + O\left(\delta u^2\right) \tag{8}$$

Using these formulas, we can get an expression for the dragged along tensor, to order $O(\delta u)$:

$$T_b^{\prime a} = \left(\delta_c^a + \delta u \frac{\partial X^a}{\partial x^c}\right)\left(\delta_b^d - \delta u \frac{\partial X^d}{\partial x^b}\right)T_d^c \tag{9}$$

$$= T_b^a + \delta u\left(\frac{\partial X^a}{\partial x^c}T_b^c - \frac{\partial X^d}{\partial x^b}T_d^a\right) \tag{10}$$

All the tensor quantities on the RHS are evaluated at point P, that is, for the unprimed coordinates x^a.

Now, for the original tensor field, we can get an expression for its actual value at Q by using a Taylor expansion to first order in δu.

$$T_{b}^{a}\left(x+\delta uX\right)=T_{b}^{a}\left(x\right)+\delta uX^{c}\frac{\partial T_{b}^{a}}{\partial x^{c}} \tag{11}$$

where again all quantities on the RHS are evaluated at P.

If we now take the difference between the 'actual' tensor at Q and the dragged along tensor at Q, and divide by δu, we get

$$\mathfrak{L}_{X}T_{b}^{a} \equiv \lim_{\delta u\to 0}\frac{T_{b}^{a}\left(x'\right)-T_{b}^{\prime a}\left(x'\right)}{\delta u} \tag{12}$$

$$= X^{c}\frac{\partial T_{b}^{a}}{\partial x^{c}}-\frac{\partial X^{a}}{\partial x^{c}}T_{b}^{c}+\frac{\partial X^{d}}{\partial x^{b}}T_{d}^{a} \tag{13}$$

This is the Lie derivative of the tensor field T_{b}^{a} with respect to the vector field X. Note that the contravariant index a contributes a term $-\frac{\partial X^{a}}{\partial x^{c}}T_{b}^{c}$ while the covariant index b contributes $+\frac{\partial X^{d}}{\partial x^{b}}T_{d}^{a}$. It's fairly obvious from the derivation (just plug in an additional first order expansion of $\frac{\partial x'^{a}}{\partial x^{c}}$ for each contravariant index and a $\frac{\partial x^{d}}{\partial x'^{b}}$ for each covariant index) that each additional index will contribute a term of the same type as that shown here. That is, in general we will get

$$\mathfrak{L}_{X}T_{cd\ldots}^{ab\ldots}=X^{e}\frac{\partial T_{cd\ldots}^{ab\ldots}}{\partial x^{e}}-\frac{\partial X^{a}}{\partial x^{e}}T_{cd\ldots}^{eb\ldots}-\frac{\partial X^{b}}{\partial x^{e}}T_{cd\ldots}^{ae\ldots}-\ldots+\frac{\partial X^{e}}{\partial x^{c}}T_{ed\ldots}^{ab\ldots}+\frac{\partial X^{e}}{\partial x^{d}}T_{ce\ldots}^{ab\ldots}+\ldots \tag{14}$$

As a simple example of this formula, we can find the Lie derivative of the Kronecker delta δ_{b}^{a}:

$$\mathfrak{L}_{X}\delta_{b}^{a} = X^{c}\partial_{c}\delta_{b}^{a}-\delta_{b}^{c}\partial_{c}X^{a}+\delta_{c}^{a}\partial_{b}X^{c} \tag{15}$$

$$= 0-\partial_{b}X^{a}+\partial_{b}X^{a} \tag{16}$$

$$= 0 \tag{17}$$

The first term is zero since δ_{b}^{a} has the same constant value in all coordinate systems. (Problem 6.1 in d'Inverno asks that you also work this derivative out 'from first principles' which presumably means you need to work out the Lie derivative term for covariant indices (since he gives only the derivation for contravariant indices in the book). Since I've already done that above, that presumably answers that part of the question.)

The Lie derivative obeys the product rule in the sense that, for any two tensor fields A and B:

$$\mathfrak{L}_X (AB) = A(\mathfrak{L}B) + (\mathfrak{L}A)B \tag{18}$$

This follows from the definition. The Lie derivative contains 3 types of terms. First there is the term $X^c \frac{\partial T}{\partial x^c}$. If $T = AB$, the ordinary product rule applies and we get

$$X^c \frac{\partial T}{\partial x^c} = X^c \frac{\partial AB}{\partial x^c} \tag{19}$$

$$= X^c \left(A\frac{\partial B}{\partial x^c} + B\frac{\partial A}{\partial x^c} \right) \tag{20}$$

For the other two types of term, we have one term for each index in the tensor of the form

$$-\frac{\partial X^a}{\partial x^e} T^{eb\ldots}_{cd\ldots} \text{ contravariant}$$

$$+\frac{\partial X^e}{\partial x^c} T^{ab\ldots}_{ed\ldots} \text{ covariant}$$

If $T^{ab\ldots}_{cd\ldots} = A^{a\ldots}_{c\ldots} B^{b\ldots}_{d\ldots}$, then these terms split into two groups. One group will have a sum over a contravariant or covariant index of A, with B just multiplied into the result, while the other group will have a sum over an index of B with A multiplied into the result. That is

$$-\frac{\partial X^a}{\partial x^e} T^{eb\ldots}_{cd\ldots} - \frac{\partial X^b}{\partial x^e} T^{ae\ldots}_{cd\ldots} - \ldots + \frac{\partial X^e}{\partial x^c} T^{ab\ldots}_{ed\ldots} + \frac{\partial X^e}{\partial x^d} T^{ab\ldots}_{ce\ldots} + \ldots = \tag{21}$$

(22)

$$-B^{b\ldots}_{d\ldots} \frac{\partial X^a}{\partial x^e} A^{e\ldots}_{c\ldots} - A^{a\ldots}_{c\ldots} \frac{\partial X^b}{\partial x^e} B^{e\ldots}_{d\ldots} - \ldots + B^{b\ldots}_{d\ldots} \frac{\partial X^e}{\partial x^c} A^{a\ldots}_{e\ldots} + A^{a\ldots}_{c\ldots} \frac{\partial X^e}{\partial x^d} B^{b\ldots}_{e\ldots} + \ldots$$

Adding up the product rule result for the first term with these other terms gives the product rule for the Lie derivative as a whole.

Combining this result with the derivative of δ^a_b above, we get

$$\mathfrak{L}_X \left(\delta^a_b T^b_a \right) = \delta^a_b \mathfrak{L}_X T^b_a + T^b_a \mathfrak{L}_X \delta^a_b \tag{23}$$

$$= \delta^a_b \mathfrak{L}_X T^b_a \tag{24}$$

However, the first quantity $\delta^a_b T^b_a$ is a contraction of the tensor T, in the sense that

$$\delta^a_b T^b_a = T^a_a \tag{25}$$

which is a scalar. So we have the result that the Lie derivative commutes with contraction:

$$\delta_b^a \mathfrak{L}_X T_a^b = \mathfrak{L}_X \left(T_a^a \right) \tag{26}$$

(Incidentally, equation 6.13 in d'Inverno has the indices the wrong way round on the LHS. The repeated indices have to occur in pairs with one upper and one lower, as shown here.)

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