

LIE DERIVATIVE: PRODUCT RULE

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Reference: d'Inverno, Ray, *Introducing Einstein's Relativity* (1992), Oxford Uni Press. - Section 6.2; Problem 6.2.

In the last post we derived the product rule for the Lie derivative. We can verify this rule in the specific case where we have a tensor product $Y^a Z_{bc}$. Using the expression in the last post for the Lie derivative of a general tensor, we have

$$\mathfrak{L}_X (Y^a Z_{bc}) = X^d \partial_d (Y^a Z_{bc}) - Y^d Z_{bc} \partial_d X^a + Y^a Z_{dc} \partial_b X^d + Y^a Z_{bd} \partial_c X^d \quad (1)$$

$$= X^d (Y^a \partial_d Z_{bc} + Z_{bc} \partial_d Y^a) - Y^d Z_{bc} \partial_d X^a + Y^a Z_{dc} \partial_b X^d + Y^a Z_{bd} \partial_c X^d \quad (2)$$

where we've used the ordinary product rule on the first term.

The Lie derivatives of each factor on its own are

$$\mathfrak{L}_X Y^a = X^d \partial_d Y^a - Y^d \partial_d X^a \quad (3)$$

$$\mathfrak{L}_X Z_{bc} = X^d \partial_d Z_{bc} + Z_{dc} \partial_b X^d + Z_{bd} \partial_c X^d \quad (4)$$

Multiply the first equation by Z_{bc} and the second by Y^a and add, then compare with the overall derivative above, and we get

$$\mathfrak{L}_X (Y^a Z_{bc}) = Z_{bc} \mathfrak{L}_X Y^a + Y^a \mathfrak{L}_X Z_{bc} \quad (5)$$