LIE DERIVATIVE: PRODUCT RULE

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Reference: d'Inverno, Ray, *Introducing Einstein's Relativity* (1992), Oxford Uni Press. - Section 6.2; Problem 6.2.

In the last post we derived the product rule for the Lie derivative. We can verify this rule in the specific case where we have a tensor product $Y^a Z_{bc}$. Using the expression in the last post for the Lie derivative of a general tensor, we have

$$\mathfrak{L}_X \left(Y^a Z_{bc}\right) = X^d \partial_d \left(Y^a Z_{bc}\right) - Y^d Z_{bc} \partial_d X^a + Y^a Z_{dc} \partial_b X^d + Y^a Z_{bd} \partial_c X^d \tag{1}$$

$$= X^d \left(Y^a \partial_d Z_{bc} + Z_{bc} \partial_d Y^a\right) - Y^d Z_{bc} \partial_d X^a + Y^a Z_{dc} \partial_b X^d + Y^a Z_{bd} \partial_c X^d \tag{2}$$

where we've used the ordinary product rule on the first term.

The Lie derivatives of each factor on its own are

$$\mathfrak{L}_X Y^a = X^d \partial_d Y^a - Y^d \partial_d X^a \tag{3}$$

$$\mathfrak{L}_X Z_{bc} = X^d \partial_d Z_{bc} + Z_{dc} \partial_b X^d + Z_{bd} \partial_c X^d \tag{4}$$

Multiply the first equation by Z_{bc} and the second by Y^a and add, then compare with the overall derivative above, and we get

$$\mathfrak{L}_X \left(Y^a Z_{bc}\right) = Z_{bc} \mathfrak{L}_X Y^a + Y^a \mathfrak{L}_X Z_{bc} \tag{5}$$