COVARIANT DERIVATIVE AND CONNECTIONS

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Reference: d'Inverno, Ray, *Introducing Einstein's Relativity* (1992), Oxford Uni Press. - Section 6.3; Problem 6.3.

The Lie derivative is one way of calculating the derivative of a tensor field in such a way that this derivative is itself a tensor. The problem solved by the Lie derivative is that we cannot define a new tensor as the difference of two other tensors evaluated at different points, since in that case the transformation between coordinate systems of this difference does not follow the equation required of a tensor.

The Lie derivative required the introduction of an auxiliary vector field which defined a congruence of curves, which in turn defined the directions along which the Lie derivative is calculated. Suppose we try to find another formula for a derivative of a vector which does not require this congruence of curves.

A general vector $\mathbf{V}$ can be written in terms of the basis vectors $\mathbf{e}_a$ in some coordinate system as

$$\mathbf{V}=V^a\mathbf{e}_a \tag{1}$$

As we vary the position, both the components of $\mathbf{V}$ and the basis vectors will, in general, vary. For example, although the basis vectors in rectangular coordinates are constant, those in polar coordinates are not. Thus if we want the derivative of $\mathbf{V}$ we have to take into account this change in the basis vectors, so we get

$$\frac{\partial\mathbf{V}}{\partial x^b}=\frac{\partial V^a}{\partial x^b}\mathbf{e}_a+V^a\frac{\partial\mathbf{e}_a}{\partial x^b} \tag{2}$$

The change in a basis vector is itself a vector, so it can be written in terms of the original set of basis vectors:

$$\frac{\partial\mathbf{e}_a}{\partial x^b}=\Gamma_{ab}^c\mathbf{e}_c \tag{3}$$

where the Γ_{ab}^c are defined by this equation, and are called the *connections*. We can use this definition to write the derivative of $\mathbf{V}$ entirely in terms of the original basis vectors:

$$\begin{aligned}
\frac{\partial \mathbf{V}}{\partial x^b} &= \frac{\partial V^a}{\partial x^b}\mathbf{e}_a + V^a\Gamma^c_{ab}\mathbf{e}_c && (4)\\
&= \frac{\partial V^a}{\partial x^b}\mathbf{e}_a + V^c\Gamma^a_{cb}\mathbf{e}_a && (5)\\
&= \left(\frac{\partial V^a}{\partial x^b} + V^c\Gamma^a_{cb}\right)\mathbf{e}_a && (6)
\end{aligned}$$

where in the second line, we swapped the dummy indices a and c. The quantity in parentheses is called the *covariant derivative* of **V** and is written in a variety of ways in different books. Two of the more common notations are

$$\nabla_b V^a \equiv V^a_{;b} \equiv \frac{\partial V^a}{\partial x^b} + V^c\Gamma^a_{cb} \tag{7}$$

We can require the covariant derivative to be a tensor, which means we can derive transformation equations for the connections Γ^a_{cb}. Since $V^a_{;b}$ is a mixed second-rank tensor, it must transform as

$$V'^a_{;b} = \frac{\partial x'^a}{\partial x^c}\frac{\partial x^d}{\partial x'^b}V^c_{;d} \tag{8}$$

Since V^a is a contravariant vector, we have

$$V'^a = \frac{\partial x'^a}{\partial x^c}V^c \tag{9}$$

Taking the derivative of this we get

$$\begin{aligned}
\frac{\partial V'^a}{\partial x'^b} &= \frac{\partial^2 x'^a}{\partial x'^b \partial x^c}V^c + \frac{\partial x'^a}{\partial x^c}\frac{\partial V^c}{\partial x'^b} && (10)\\
&= \frac{\partial x^d}{\partial x'^b}\frac{\partial^2 x'^a}{\partial x^d \partial x^c}V^c + \frac{\partial x^d}{\partial x'^b}\frac{\partial x'^a}{\partial x^c}\frac{\partial V^c}{\partial x^d} && (11)
\end{aligned}$$

For the second term in 7, we have

$$V'^c\Gamma'^a_{cb} = \frac{\partial x'^c}{\partial x^d}V^d\Gamma'^a_{cb} \tag{12}$$

Summing these last two results and requiring they give 8 gives

$$\frac{\partial x^d}{\partial x'^b}\frac{\partial^2 x'^a}{\partial x^d \partial x^c}V^c + \frac{\partial x^d}{\partial x'^b}\frac{\partial x'^a}{\partial x^c}\frac{\partial V^c}{\partial x^d} + \frac{\partial x'^c}{\partial x^d}V^d\Gamma'^a_{cb} = \frac{\partial x'^a}{\partial x^c}\frac{\partial x^d}{\partial x'^b}V^c_{;d} \tag{13}$$

$$= \frac{\partial x'^a}{\partial x^c}\frac{\partial x^d}{\partial x'^b}\left(\frac{\partial V^c}{\partial x^d} + V^e\Gamma^c_{ed}\right) \tag{14}$$

$$\frac{\partial x^d}{\partial x'^b}\frac{\partial^2 x'^a}{\partial x^d \partial x^c}V^c + \frac{\partial x'^c}{\partial x^d}V^d\Gamma'^a_{cb} = \frac{\partial x'^a}{\partial x^c}\frac{\partial x^d}{\partial x'^b}V^e\Gamma^c_{ed} \tag{15}$$

$$\frac{\partial x^d}{\partial x'^b}\frac{\partial^2 x'^a}{\partial x^d \partial x^e}V^e + \frac{\partial x'^c}{\partial x^e}V^e\Gamma'^a_{cb} = \frac{\partial x'^a}{\partial x^c}\frac{\partial x^d}{\partial x'^b}V^e\Gamma^c_{ed} \tag{16}$$

$$\frac{\partial x^d}{\partial x'^b}\frac{\partial^2 x'^a}{\partial x^d \partial x^e} + \frac{\partial x'^c}{\partial x^e}\Gamma'^a_{cb} = \frac{\partial x'^a}{\partial x^c}\frac{\partial x^d}{\partial x'^b}\Gamma^c_{ed} \tag{17}$$

In the fourth line we relabelled the dummy index on V^c and V^d to give all the Vs the same index. In the last line, we can cancel off V^e since this equation must be true for all vectors, which means the coefficients of each component V^e must be equal.

To isolate Γ'^a_{cb} we can multiply both sides of this equation by $\partial x^e/\partial x'^f$ and sum over e:

$$\frac{\partial x^e}{\partial x'^f}\frac{\partial x^d}{\partial x'^b}\frac{\partial^2 x'^a}{\partial x^d \partial x^e} + \frac{\partial x^e}{\partial x'^f}\frac{\partial x'^c}{\partial x^e}\Gamma'^a_{cb} = \frac{\partial x^e}{\partial x'^f}\frac{\partial x'^a}{\partial x^c}\frac{\partial x^d}{\partial x'^b}\Gamma^c_{ed} \tag{18}$$

Since

$$\frac{\partial x^e}{\partial x'^f}\frac{\partial x'^c}{\partial x^e} = \delta^c_f \tag{19}$$

we get

$$\Gamma'^a_{fb} = \frac{\partial x^e}{\partial x'^f}\frac{\partial x'^a}{\partial x^c}\frac{\partial x^d}{\partial x'^b}\Gamma^c_{ed} - \frac{\partial x^e}{\partial x'^f}\frac{\partial x^d}{\partial x'^b}\frac{\partial^2 x'^a}{\partial x^d \partial x^e} \tag{20}$$

The second term can be compressed a little by the calculation:

$$\delta^a_b = \frac{\partial x'^a}{\partial x^d}\frac{\partial x^d}{\partial x'^b} \tag{21}$$

$$\frac{\partial \delta^a_b}{\partial x'^f} = \frac{\partial^2 x'^a}{\partial x^d \partial x'^f}\frac{\partial x^d}{\partial x'^b} + \frac{\partial x'^a}{\partial x^d}\frac{\partial^2 x^d}{\partial x'^b \partial x'^f} \tag{22}$$

$$= 0 \tag{23}$$

since $\frac{\partial \delta^a_b}{\partial x'^f} = 0$, as δ^a_b is a constant tensor.

Thus we get

$$\frac{\partial^2 x'^a}{\partial x^d \partial x'^f}\frac{\partial x^d}{\partial x'^b} = -\frac{\partial x'^a}{\partial x^d}\frac{\partial^2 x^d}{\partial x'^b \partial x'^f} \tag{24}$$

$$\frac{\partial^2 x'^a}{\partial x^d \partial x^e}\frac{\partial x^e}{\partial x'^f}\frac{\partial x^d}{\partial x'^b} = -\frac{\partial x'^a}{\partial x^d}\frac{\partial^2 x^d}{\partial x'^b \partial x'^f} \tag{25}$$

Substituting this into 20 we get

$$\Gamma'^a_{fb} = \frac{\partial x^e}{\partial x'^f}\frac{\partial x'^a}{\partial x^c}\frac{\partial x^d}{\partial x'^b}\Gamma^c_{ed} + \frac{\partial x'^a}{\partial x^d}\frac{\partial^2 x^d}{\partial x'^b \partial x'^f} \tag{26}$$

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