COVARIANT DERIVATIVE OF HIGHER RANK TENSORS

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Reference: d'Inverno, Ray, *Introducing Einstein's Relativity* (1992), Oxford Uni Press. - Section 6.3; Problem 6.4.

We've seen the covariant derivative for the contravariant and covariant vector, but what about higher order tensors? For the special case where the higher order tensor can be written as a product of vectors, we can impose the product rule in the same way we did to derive the derivative of a covariant vector. For example, if we have a tensor $T_b^a = X^a Y_b$, then we can require

$$\begin{aligned} T_{b;c}^a &= \left(X^a Y_b\right)_{;c} && (1)\\ &= X^a Y_{b;c} + X_{;c}^a Y_b && (2) \end{aligned}$$

We can now plug in the expressions we have for each of these derivatives (see earlier posts) and we get

$$\begin{aligned} T_{b;c}^a &= X^a\left(\partial_c Y_b - Y_d \Gamma_{bc}^d\right) + Y_b\left(\partial_c X^a + X^d \Gamma_{dc}^a\right) && (3)\\ &= X^a \partial_c Y_b + Y_b \partial_c X^a - X^a Y_d \Gamma_{bc}^d + Y_b X^d \Gamma_{dc}^a && (4)\\ &= \partial_c\left(X^a Y_b\right) - X^a Y_d \Gamma_{bc}^d + X^d Y_b \Gamma_{dc}^a && (5)\\ &= \partial_c T_b^a - T_d^a \Gamma_{bc}^d + T_b^d \Gamma_{dc}^a && (6) \end{aligned}$$

We see that we get a term of form $-T_d^a \Gamma_{bc}^d$ for the covariant index b, and a term $+T_b^d \Gamma_{dc}^a$ for the contravariant index a. It's fairly easy to see that if we have a tensor of any rank that can be split up into a product of vectors, such as

$$T_{cd\ldots}^{ab\ldots} = A^a B^b C_c D_d \ldots \qquad (7)$$

then we get one term of the corresponding kind for each vector in the product.

However, not every higher rank tensor can be expressed as a product of vectors. As far as I can tell, the covariant derivative of a general higher rank tensor is simply *defined* so that it contains terms as specified here. That is, for a tensor $T_{cd\ldots}^{ab\ldots}$, even if it *can't* be expressed as a product of vectors, its covariant derivative is defined to be

$$T^{ab\ldots}_{cd\ldots;e} = \partial_e T^{ab\ldots}_{cd\ldots} + T^{fb\ldots}_{cd\ldots}\Gamma^a_{fe} + T^{af\ldots}_{cd\ldots}\Gamma^b_{fe} + \ldots - T^{ab\ldots}_{fd\ldots}\Gamma^f_{ce} - T^{ab\ldots}_{cf\ldots}\Gamma^f_{de} - \ldots \tag{8}$$

As a simple example of this formula, the covariant derivative of the Kronecker delta δ^a_b is

$$\begin{aligned}\delta^a_{b;c} &= \partial_c\delta^a_b + \delta^d_b\Gamma^a_{dc} - \delta^a_d\Gamma^d_{bc} && (9)\\ &= 0 + \Gamma^a_{bc} - \Gamma^a_{bc} && (10)\\ &= 0 && (11)\end{aligned}$$

Thus its derivative is zero which means that the covariant derivative, like the Lie derivative, commutes with contraction:

$$\begin{aligned}\left(\delta^a_b T^b_a\right)_{;c} &= \delta^a_{b;c}T^b_a + \delta^a_b T^b_{a;c} && (12)\\ &= T^a_{a;c} && (13)\end{aligned}$$

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