COVARIANT DERIVATIVE OF COVARIANT VECTOR

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Reference: d'Inverno, Ray, *Introducing Einstein's Relativity* (1992), Oxford Uni Press. - Section 6.3; Problem 6.5.

In the last post we derived the covariant derivative of a contravariant vector and found that

$$V^a_{;b} \equiv \frac{\partial V^a}{\partial x^b} + V^c \Gamma^a_{cb} \tag{1}$$

Unlike with the Lie derivative, we can't derive the corresponding expression for a covariant vector by considering coordinate transformations, since the covariant derivative doesn't rely on these transformations. What is done instead is to require that the covariant derivative obeys the product rule. To show how this works, we first need the covariant derivative of a scalar. Since the ordinary derivative of a scalar already behaves as a rank-one covariant tensor, we can just take the covariant derivative to be the same as an ordinary derivative:

$$\phi_{;a} \equiv \frac{\partial \phi}{\partial x^a} \tag{2}$$

Now we consider the quantity $X_a Y^a$ for two arbitrary vectors X and Y. Because of the implied sum, this is a scalar quantity, so we get

$$\begin{aligned} (X_a Y^a)_{;b} &= \partial_b (X_a Y^a) & (3) \\ &= Y^a \partial_b X_a + X_a \partial_b Y^a & (4) \end{aligned}$$

Now if we require the covariant derivative to obey the product rule we must also have

$$(X_a Y^a)_{;b} = Y^a X_{a;b} + X_a Y^a_{;b} \tag{5}$$

We already have an expression for $Y^a_{;b} = \partial_b Y^a + Y^c \Gamma^a_{cb}$ so if we plug this in and require the result equal to the ordinary derivative above, we get

$$\begin{aligned} Y^a \partial_b X_a + X_a \partial_b Y^a &= Y^a X_{a;b} + X_a \left(\partial_b Y^a + Y^c \Gamma^a_{cb}\right) & (6) \\ Y^a \partial_b X_a &= Y^a X_{a;b} + X_a Y^c \Gamma^a_{cb} & (7) \\ Y^a \left(\partial_b X_a - X_c \Gamma^c_{ab}\right) &= Y^a X_{a;b} & (8) \end{aligned}$$

Since this must be true for any vector Y, the coefficients of Y^a on each side must be equal and we get

$$X_{a;b} = \partial_b X_a - X_c \Gamma^c_{ab} \tag{9}$$

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