LIE DERIVATIVE IN TERMS OF THE COVARIANT DERIVATIVE

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Reference: d'Inverno, Ray, *Introducing Einstein's Relativity* (1992), Oxford Uni Press. - Section 6.3; Problem 6.6.

The covariant derivative of a general tensor is

$$T_{cd\ldots;e}^{ab\ldots}=\partial_{e}T_{cd\ldots}^{ab\ldots}+T_{cd\ldots}^{fb\ldots}\Gamma_{fe}^{a}+T_{cd\ldots}^{af\ldots}\Gamma_{fe}^{b}+\ldots-T_{fd\ldots}^{ab\ldots}\Gamma_{ce}^{f}-T_{cf\ldots}^{ab\ldots}\Gamma_{de}^{f}-\ldots \tag{1}$$

where the Γs are the connections, that are defined by the way the basis vectors change when their derivatives are taken:

$$\frac{\partial\mathbf{e}_{a}}{\partial x^{b}}=\Gamma_{ab}^{c}\mathbf{e}_{c} \tag{2}$$

In general, the Γ_{ab}^{c} don't possess any special symmetries, but a special subset of connections satisfy the condition

$$\Gamma_{ab}^{c}=\Gamma_{ba}^{c} \tag{3}$$

and most of general relativity is concerned with such symmetric connections.

Under this condition, we can derive a relation between the Lie and covariant derivatives. The Lie derivative is

$$\mathfrak{L}_{X}T_{cd\ldots}^{ab\ldots}=X^{e}\partial_{e}T_{cd\ldots}^{ab\ldots}-\partial_{e}X^{a}T_{cd\ldots}^{eb\ldots}-\partial_{e}X^{b}T_{cd\ldots}^{ae\ldots}-\ldots+\partial_{c}X^{e}T_{ed\ldots}^{ab\ldots}+\partial_{d}X^{e}T_{ce\ldots}^{ab\ldots}+\ldots \tag{4}$$

With symmetric connections, we can replace every partial derivative in this equation with a covariant derivative. Doing this, we get

$$\mathfrak{L}_{X}T_{cd\ldots}^{ab\ldots}=X^{e}\left(\partial_{e}T_{cd\ldots}^{ab\ldots}+T_{cd\ldots}^{fb\ldots}\Gamma_{fe}^{a}+T_{cd\ldots}^{af\ldots}\Gamma_{fe}^{b}+\ldots-T_{fd\ldots}^{ab\ldots}\Gamma_{ce}^{f}-T_{cf\ldots}^{ab\ldots}\Gamma_{de}^{f}-\ldots\right) \tag{5}$$

$$-\left(\partial_{e}X^{a}+X^{f}\Gamma_{fe}^{a}\right)T_{cd\ldots}^{eb\ldots}-\left(\partial_{e}X^{b}+X^{f}\Gamma_{fe}^{b}\right)T_{cd\ldots}^{ae\ldots}-\ldots \tag{6}$$

$$+\left(\partial_{c}X^{e}+X^{f}\Gamma_{fc}^{e}\right)T_{ed\ldots}^{ab\ldots}+\left(\partial_{d}X^{e}+X^{f}\Gamma_{fd}^{e}\right)T_{ce\ldots}^{ab\ldots}+\ldots \tag{7}$$

The first term in the first line here matches the first term in 4. The first part in each of the terms on the second and third lines matches the corresponding derivative in 4. The terms involving the connections all cancel in pairs if we assume symmetric connections. For example, the second term in the first line is $X^{e}T_{cd\ldots}^{fb\ldots}\Gamma_{fe}^{a}$ and the first connection term in the second line is $-X^{f}\Gamma_{fe}^{a}T_{cd\ldots}^{eb\ldots}$. If we apply symmetry to the first of these we get

$$X^{e}T_{cd\ldots}^{fb\ldots}\Gamma_{fe}^{a} = X^{e}T_{cd\ldots}^{fb\ldots}\Gamma_{ef}^{a} \tag{8}$$

$$= X^{f}T_{cd\ldots}^{eb\ldots}\Gamma_{fe}^{a} \tag{9}$$

where in the last line we've merely relabelled the dummy indices e and f. Thus this term cancels the other term $-X^{f}\Gamma_{fe}^{a}T_{cd\ldots}^{eb\ldots}$. All the other terms involving connections cancel in the same way. Thus we can write

$$\mathfrak{L}_{X}T_{cd\ldots}^{ab\ldots} = X^{e}T_{cd\ldots;e}^{ab\ldots} - X_{;e}^{a}T_{cd\ldots}^{eb\ldots} - X_{;e}^{b}T_{cd\ldots}^{ae\ldots} - \ldots + X_{;c}^{e}T_{ed\ldots}^{ab\ldots} + X_{;d}^{e}T_{ce\ldots}^{ab\ldots} + \ldots \tag{10}$$

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