# AFFINE GEODESIC AND AFFINE PARAMETER

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Reference: d'Inverno, Ray, *Introducing Einstein's Relativity* (1992), Oxford Uni Press. - Section 6.4; Problem 6.8.

Parallel transport of a vector $V^a$ along a curve parametrized by a variable $u$ occurs if we can satisfy the condition

$$\nabla_X V^a = \frac{dx^b}{du}\left(\frac{\partial V^a}{\partial x^b}+V^c\Gamma_{cb}^a\right)=0 \tag{1}$$

where $X^b=\frac{dx^b}{du}$ is the tangent to the curve.

Parallel transport is usually defined as moving the vector (or tensor) along the curve without changing it, either in magnitude or direction (or in any of the dimensions, if we're dealing with a higher-rank tensor). This condition is expressed by requiring the total derivative of the tangent vector to be zero, as the above equation specifies.

If we relax this condition a bit, and require only that the vector has the same direction, but not necessarily the same magnitude, as it propagates along the curve, then the condition becomes instead

$$\nabla_X V^a = \lambda(u)V^a \tag{2}$$

where $\lambda(u)$ is some scalar function of $u$. This is easier to see if we consider a 3-d example in flat space. If we have a 3-d vector field $\mathbf{v}=a\hat{\mathbf{x}}+b\hat{\mathbf{y}}+c\hat{\mathbf{z}}$ where $a,b,c$ are constants (so the vector field is constant over all space), then if we propagate this vector along any curve, it remains the same, so it satisfies the condition for parallel transport along any curve.

Now suppose we multiply this field by some scalar function $f(\mathbf{r})$, where $\mathbf{r}$ is the position vector. We then get

$$\mathbf{v}'=af(\mathbf{r})\hat{\mathbf{x}}+bf(\mathbf{r})\hat{\mathbf{y}}+cf(\mathbf{r})\hat{\mathbf{z}} \tag{3}$$

If we consider a particular curve $\mathbf{r}(u)$ then we can write $f$ as a function of $u$ along this curve, so that along the curve we have

$$\mathbf{v}'=af(u)\hat{\mathbf{x}}+bf(u)\hat{\mathbf{y}}+cf(u)\hat{\mathbf{z}} \tag{4}$$

Since $\mathbf{v}'$ is merely multiplied by a scalar as we move along the curve, all instances of $\mathbf{v}'$are parallel everywhere on the curve. If we take the derivative with respect to $u$, we get

$$\frac{d\mathbf{v}'}{du}=af'(u)\hat{\mathbf{x}}+bf'(u)\hat{\mathbf{y}}+cf'(u)\hat{\mathbf{z}} \tag{5}$$

or, if we set $\frac{f'(u)}{f(u)}=\lambda(u)$ we get

$$\frac{d\mathbf{v}'}{du}=\lambda(u)\mathbf{v}' \tag{6}$$

(OK, we *could* have just required that the vector field at each point on the curve is itself is a scalar multiple of the vector at a given point, but the condition here is a bit more general in that it doesn't make reference to any particular point on the curve.)

Now we can look at a very special case. If the vector being transported along the curve is the tangent vector itself, then that tangent vector will be propagated parallel to itself if it satisfies the above condition, that is

$$\nabla_X\frac{dx^a}{du}=\lambda(u)\frac{dx^a}{du} \tag{7}$$

It's worth clearing up a bit of confusion that arises (at least for me) in d'Inverno's section 6.4. He initially says that a tensor $T^{ab\ldots}_{cd\ldots}$ is parallely propagated along a curve if $\nabla_X T^{ab\ldots}_{cd\ldots}=0$, which is fine. However, he then says the tangent vector is parallely propagated if it satisfies 7, which is *not* the same thing. Equation 7 means only that the tangent vector remains parallel to itself, but does not guarantee that it retains the same magnitude; that happens only if $\lambda(u)=0$ everywhere on the curve. The terminology is confusing, since a vector *is* actually propagated parallel to itself in both cases, but he defines (in his equation 6.32) the particular case where $\lambda=0$ as parallel propagation (that is, the tensor is propagated unchanged in direction or magnitude), and then goes on to use the more general definition 7 (where only the direction is unchanged) in discussing tangent vectors. He does return to the case $\lambda=0$ in equation 6.36.

If a curve on which the tangent vector satisfies 7 can be found, such a curve is called an *affine geodesic*. Further, if the curve can be parametrized in such a way $\lambda(u)=0$ then the parameter $u$ is called an affine parameter. An affine parameter is often given the symbol $s$ instead of $u$.

It's important to note that $\lambda$ is *not* a parameter that defines a curve. It serves only to give the proportionality between the total derivative of a vector and the vector itself as you move along the curve.

Starting from 7, we have

$$\nabla_X \frac{dx^a}{du} = \frac{dx^b}{du}\left(\frac{\partial}{\partial x^b}\frac{dx^a}{du} + \frac{dx^c}{du}\Gamma^a_{cb}\right) \tag{8}$$

$$= \frac{d^2x^a}{du^2} + \frac{dx^b}{du}\frac{dx^c}{du}\Gamma^a_{cb} \tag{9}$$

using the chain rule to condense the first term. Thus we get the relation

$$\frac{d^2x^a}{du^2} + \frac{dx^b}{du}\frac{dx^c}{du}\Gamma^a_{cb} = \lambda(u)\frac{dx^a}{du} \tag{10}$$

In the case of a curve parametrized by an affine parameter, we get

$$\frac{d^2x^a}{du^2} + \frac{dx^b}{du}\frac{dx^c}{du}\Gamma^a_{cb} = 0 \tag{11}$$

## PINGBACKS

Pingback: Affine parameter transformation