AFFINE PARAMETER TRANSFORMATION

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Reference: d'Inverno, Ray, *Introducing Einstein's Relativity* (1992), Oxford Uni Press. - Section 6.4; Problem 6.9.

An affine parameter s is a parameter that can be used to define a curve in such a way that the tangent vector to the curve remains constant as it is transported along the curve using parallel transport. This condition is expressed as

$$\frac{d^2x^a}{ds^2}+\frac{dx^b}{ds}\frac{dx^c}{ds}\Gamma_{cb}^a=0 \tag{1}$$

where Γ_{cb}^a is the affine connection.

If we define a transformation of the parameter by

$$s\rightarrow\sigma(s) \tag{2}$$

where σ is some differentiable function of s, we can investigate the conditions on σ such that it too is an affine parameter.

The terms in the equation above then transform as

$$\frac{d^2x^a}{ds^2} = \frac{d}{ds}\left(\frac{dx^a}{ds}\right) \tag{3}$$

$$= \frac{d}{ds}\left(\frac{dx^a}{d\sigma}\sigma'\right) \tag{4}$$

$$= \frac{d^2x^a}{d\sigma^2}\left(\sigma'\right)^2+\frac{dx^a}{d\sigma}\sigma'' \tag{5}$$

$$\frac{dx^b}{ds} = \frac{dx^b}{d\sigma}\sigma' \tag{6}$$

Plugging these into the original equation we get

$$\frac{d^2x^a}{d\sigma^2}\left(\sigma'\right)^2+\frac{dx^a}{d\sigma}\sigma''+\frac{dx^b}{d\sigma}\frac{dx^c}{d\sigma}\left(\sigma'\right)^2\Gamma_{cb}^a=0 \tag{7}$$

If σ is to be an affine parameter, then we must have

$$\frac{d^2x^a}{d\sigma^2} + \frac{dx^b}{d\sigma}\frac{dx^c}{d\sigma}\Gamma^a_{cb} = 0 \tag{8}$$

which leaves us with

$$\frac{dx^a}{d\sigma}\sigma'' = 0 \tag{9}$$

Since the tangent vector $\frac{dx^a}{d\sigma}$ will not be zero in general, we must have

$$\sigma'' = 0 \tag{10}$$

from which we get by direct integration

$$\sigma = \alpha s + \beta \tag{11}$$

for some constants α and β.