

AFFINE PARAMETER TRANSFORMATION

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Reference: d'Inverno, Ray, *Introducing Einstein's Relativity* (1992), Oxford Uni Press. - Section 6.4; Problem 6.9.

An affine parameter s is a parameter that can be used to define a curve in such a way that the tangent vector to the curve remains constant as it is transported along the curve using parallel transport. This condition is expressed as

$$\frac{d^2x^a}{ds^2} + \frac{dx^b}{ds} \frac{dx^c}{ds} \Gamma_{cb}^a = 0 \quad (1)$$

where Γ_{cb}^a is the affine connection.

If we define a transformation of the parameter by

$$s \rightarrow \sigma(s) \quad (2)$$

where σ is some differentiable function of s , we can investigate the conditions on σ such that it too is an affine parameter.

The terms in the equation above then transform as

$$\frac{d^2x^a}{ds^2} = \frac{d}{ds} \left(\frac{dx^a}{ds} \right) \quad (3)$$

$$= \frac{d}{ds} \left(\frac{dx^a}{d\sigma} \sigma' \right) \quad (4)$$

$$= \frac{d^2x^a}{d\sigma^2} (\sigma')^2 + \frac{dx^a}{d\sigma} \sigma'' \quad (5)$$

$$\frac{dx^b}{ds} = \frac{dx^b}{d\sigma} \sigma' \quad (6)$$

Plugging these into the original equation we get

$$\frac{d^2x^a}{d\sigma^2} (\sigma')^2 + \frac{dx^a}{d\sigma} \sigma'' + \frac{dx^b}{d\sigma} \frac{dx^c}{d\sigma} (\sigma')^2 \Gamma_{cb}^a = 0 \quad (7)$$

If σ is to be an affine parameter, then we must have

$$\frac{d^2 x^a}{d\sigma^2} + \frac{dx^b}{d\sigma} \frac{dx^c}{d\sigma} \Gamma_{cb}^a = 0 \quad (8)$$

which leaves us with

$$\frac{dx^a}{d\sigma} \sigma'' = 0 \quad (9)$$

Since the tangent vector $\frac{dx^a}{d\sigma}$ will not be zero in general, we must have

$$\sigma'' = 0 \quad (10)$$

from which we get by direct integration

$$\sigma = \alpha s + \beta \quad (11)$$

for some constants α and β .