

## AFFINE PARAMETER TRANSFORMATION

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Reference: d'Inverno, Ray, *Introducing Einstein's Relativity* (1992), Oxford Uni Press. - Section 6.4; Problem 6.9.

An affine parameter  $s$  is a parameter that can be used to define a curve in such a way that the tangent vector to the curve remains constant as it is transported along the curve using parallel transport. This condition is expressed as

$$(0.1) \quad \frac{d^2 x^a}{ds^2} + \frac{dx^b}{ds} \frac{dx^c}{ds} \Gamma_{cb}^a = 0$$

where  $\Gamma_{cb}^a$  is the affine connection.

If we define a transformation of the parameter by

$$(0.2) \quad s \rightarrow \sigma(s)$$

where  $\sigma$  is some differentiable function of  $s$ , we can investigate the conditions on  $\sigma$  such that it too is an affine parameter.

The terms in the equation above then transform as

$$(0.3) \quad \frac{d^2 x^a}{ds^2} = \frac{d}{ds} \left( \frac{dx^a}{ds} \right)$$

$$(0.4) \quad = \frac{d}{ds} \left( \frac{dx^a}{d\sigma} \sigma' \right)$$

$$(0.5) \quad = \frac{d^2 x^a}{d\sigma^2} (\sigma')^2 + \frac{dx^a}{d\sigma} \sigma''$$

$$(0.6) \quad \frac{dx^b}{ds} = \frac{dx^b}{d\sigma} \sigma'$$

Plugging these into the original equation we get

$$(0.7) \quad \frac{d^2 x^a}{d\sigma^2} (\sigma')^2 + \frac{dx^a}{d\sigma} \sigma'' + \frac{dx^b}{d\sigma} \frac{dx^c}{d\sigma} (\sigma')^2 \Gamma_{cb}^a = 0$$

If  $\sigma$  is to be an affine parameter, then we must have

$$(0.8) \quad \frac{d^2 x^a}{d\sigma^2} + \frac{dx^b}{d\sigma} \frac{dx^c}{d\sigma} \Gamma_{cb}^a = 0$$

which leaves us with

$$(0.9) \quad \frac{dx^a}{d\sigma} \sigma'' = 0$$

Since the tangent vector  $\frac{dx^a}{d\sigma}$  will not be zero in general, we must have

$$(0.10) \quad \sigma'' = 0$$

from which we get by direct integration

$$(0.11) \quad \sigma = \alpha s + \beta$$

for some constants  $\alpha$  and  $\beta$ .