RIEMANN TENSOR - COMMUTATOR OF RANK 2 TENSOR

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Reference: d'Inverno, Ray, *Introducing Einstein's Relativity* (1992), Oxford Uni Press. - Section 6.5; Problem 6.10.

The covariant derivative of a contravariant vector is defined as

$$\nabla_b V^a \equiv V^a_{;b} \equiv \frac{\partial V^a}{\partial x^b} + V^c \Gamma^a_{cb} \tag{1}$$

This is generalized to the covariant derivative of a higher-rank tensor by the formula

$$T^{ab\ldots}_{cd\ldots;e} = \partial_e T^{ab\ldots}_{cd\ldots} + T^{fb\ldots}_{cd\ldots}\Gamma^a_{fe} + T^{af\ldots}_{cd\ldots}\Gamma^b_{fe} + \ldots - T^{ab\ldots}_{fd\ldots}\Gamma^f_{ce} - T^{ab\ldots}_{cf\ldots}\Gamma^f_{de} - \ldots \tag{2}$$

Ordinary partial derivatives, for a continuously differentiable function $f(x^a)$, are commutative, that is

$$\frac{\partial}{\partial x^b}\left(\frac{\partial f}{\partial x^a}\right) = \frac{\partial}{\partial x^a}\left(\frac{\partial f}{\partial x^b}\right) \tag{3}$$

The covariant derivative, however, is not in general commutative, as we can verify by direct calculation. We want to find

$$X^a{}_{b;c;d} - X^a{}_{b;d;c} \tag{4}$$

which is known as the commutator of the tensor $X^a{}_b$. For the first term, we get, using 2

$$\begin{aligned}
X^a{}_{b;c;d} &= \partial_d X^a{}_{b;c} + X^e{}_{b:c}\Gamma^a_{ed} - X^a{}_{e:c}\Gamma^e_{bd} - X^a{}_{b:e}\Gamma^e_{cd} && (5)\\
&= \partial_d\left(\partial_c X^a{}_b + X^e{}_b\Gamma^a_{ec} - X^a{}_e\Gamma^e_{bc}\right) + && (6)\\
&\Gamma^a_{ed}\left(\partial_c X^e{}_b + X^f{}_b\Gamma^e_{fc} - X^e{}_f\Gamma^f_{bc}\right) - && (7)\\
&\Gamma^e_{bd}\left(\partial_c X^a{}_e + X^f{}_e\Gamma^a_{fc} - X^a{}_f\Gamma^f_{ec}\right) - && (8)\\
&\Gamma^e_{cd}\left(\partial_e X^a{}_b + X^f{}_b\Gamma^a_{fe} - X^a{}_f\Gamma^f_{be}\right) && (9)
\end{aligned}$$

The other term can be obtained by simply swapping the indices c and d:

$$
\begin{aligned}
X^{a}{}_{b;d;c} &= \partial_c X^{a}{}_{b;d} + X^{e}{}_{b:d}\Gamma^{a}_{ec} - X^{a}{}_{e:d}\Gamma^{e}_{bc} - X^{a}{}_{b:e}\Gamma^{e}_{dc} \qquad (10)\\
&= \partial_c \left(\partial_d X^{a}{}_{b} + X^{e}{}_{b}\Gamma^{a}_{ed} - X^{a}{}_{e}\Gamma^{e}_{bd}\right) + \qquad (11)\\
&\Gamma^{a}_{ec}\left(\partial_d X^{e}{}_{b} + X^{f}{}_{b}\Gamma^{e}_{fd} - X^{e}{}_{f}\Gamma^{f}_{bd}\right) - \qquad (12)\\
&\Gamma^{e}_{bc}\left(\partial_d X^{a}{}_{e} + X^{f}{}_{e}\Gamma^{a}_{fd} - X^{a}{}_{f}\Gamma^{f}_{ed}\right) - \qquad (13)\\
&\Gamma^{e}_{dc}\left(\partial_e X^{a}{}_{b} + X^{f}{}_{b}\Gamma^{a}_{fe} - X^{a}{}_{f}\Gamma^{f}_{be}\right) \qquad (14)
\end{aligned}
$$

Now we need to take the difference. Assuming the ordinary partial derivatives commute and using the product rule, we get

$$
\begin{aligned}
X^{a}{}_{b;c;d} - X^{a}{}_{b;d;c} &= X^{e}{}_{b}\left(\partial_d\Gamma^{a}_{ec} - \partial_c\Gamma^{a}_{ed}\right) - X^{a}{}_{e}\left(\partial_d\Gamma^{e}_{bc} - \partial_c\Gamma^{e}_{bd}\right) + \qquad (15)\\
&X^{f}{}_{b}\left(\Gamma^{a}_{ed}\Gamma^{e}_{fc} - \Gamma^{a}_{ec}\Gamma^{e}_{fd}\right) - X^{e}{}_{f}\left(\Gamma^{a}_{ed}\Gamma^{f}_{bc} - \Gamma^{a}_{ec}\Gamma^{f}_{bd}\right) - \qquad (16)\\
&X^{f}{}_{e}\left(\Gamma^{e}_{bd}\Gamma^{a}_{fc} - \Gamma^{e}_{bc}\Gamma^{a}_{fd}\right) + X^{a}{}_{f}\left(\Gamma^{e}_{bd}\Gamma^{f}_{ec} - \Gamma^{e}_{bc}\Gamma^{f}_{ed}\right) - \qquad (17)\\
&\left(\partial_e X^{a}{}_{b} + X^{f}{}_{b}\Gamma^{a}_{fe} - X^{a}{}_{f}\Gamma^{f}_{be}\right)\left(\Gamma^{e}_{cd} - \Gamma^{e}_{dc}\right) \qquad (18)
\end{aligned}
$$

We can now swap the indices e and f in the first term in the third line (since they are both dummy indices) to get

$$
X^{f}{}_{e}\left(\Gamma^{e}_{bd}\Gamma^{a}_{fc} - \Gamma^{e}_{bc}\Gamma^{a}_{fd}\right) = X^{e}{}_{f}\left(\Gamma^{f}_{bd}\Gamma^{a}_{ec} - \Gamma^{f}_{bc}\Gamma^{a}_{ed}\right) \tag{19}
$$

We can now see that this term cancels the last term on the second line. If we also assume that the affine connections are symmetric, so that

$$
\Gamma^{e}_{cd} = \Gamma^{e}_{dc} \tag{20}
$$

then the last line disappears and we are left with

$$
\begin{aligned}
X^{a}{}_{b;c;d} - X^{a}{}_{b;d;c} &= X^{e}{}_{b}\left(\partial_d\Gamma^{a}_{ec} - \partial_c\Gamma^{a}_{ed}\right) - X^{a}{}_{e}\left(\partial_d\Gamma^{e}_{bc} - \partial_c\Gamma^{e}_{bd}\right) + \qquad (21)\\
&X^{e}{}_{b}\left(\Gamma^{a}_{fd}\Gamma^{f}_{ec} - \Gamma^{a}_{fc}\Gamma^{f}_{ed}\right) - X^{a}{}_{e}\left(\Gamma^{f}_{bc}\Gamma^{e}_{fd} - \Gamma^{f}_{bd}\Gamma^{e}_{fc}\right) \qquad (22)\\
&= X^{e}{}_{b}\left(\partial_d\Gamma^{a}_{ec} - \partial_c\Gamma^{a}_{ed} + \Gamma^{a}_{fd}\Gamma^{f}_{ec} - \Gamma^{a}_{fc}\Gamma^{f}_{ed}\right) - \qquad (23)\\
&X^{a}{}_{e}\left(\partial_d\Gamma^{e}_{bc} - \partial_c\Gamma^{e}_{bd} + \Gamma^{f}_{bc}\Gamma^{e}_{fd} - \Gamma^{f}_{bd}\Gamma^{e}_{fc}\right) \qquad (24)
\end{aligned}
$$

where again we have swapped e and f in the second line.

The two terms in parentheses have the same form, and they are known as the *Riemann tensor* or *curvature tensor,* defined by

$$R^a_{\ edc} \equiv \partial_d \Gamma^a_{ec} - \partial_c \Gamma^a_{ed} + \Gamma^a_{fd} \Gamma^f_{ec} - \Gamma^a_{fc} \Gamma^f_{ed} \tag{25}$$

In terms of the Riemann tensor, we get for the commutator:

$$X^a_{\ b;c;d} - X^a_{\ b;d;c} = X^e_{\ b} R^a_{\ edc} - X^a_{\ e} R^e_{\ bdc} \tag{26}$$

This is actually the same result as given in d'Inverno's problem 6.10, with c and d swapped around; I just took the original covariant derivatives in the opposite order to d'Inverno and can't be bothered going through the whole derivation again to change it.

PINGBACKS

Pingback: Riemann tensor and covariant contraction