

## RIEMANN TENSOR - COMMUTATOR OF RANK 2 TENSOR

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Reference: d'Inverno, Ray, *Introducing Einstein's Relativity* (1992), Oxford Uni Press. - Section 6.5; Problem 6.10.

The covariant derivative of a contravariant vector is defined as

$$(0.1) \quad \nabla_b V^a \equiv V^a_{;b} \equiv \frac{\partial V^a}{\partial x^b} + V^c \Gamma_{cb}^a$$

This is generalized to the covariant derivative of a higher-rank tensor by the formula

$$(0.2) \quad T_{cd...;e}^{ab...} = \partial_e T_{cd...}^{ab...} + T_{cd...}^{fb...} \Gamma_{fe}^a + T_{cd...}^{af...} \Gamma_{fe}^b + \dots - T_{fd...}^{ab...} \Gamma_{ce}^f - T_{cf...}^{ab...} \Gamma_{de}^f - \dots$$

Ordinary partial derivatives, for a continuously differentiable function  $f(x^a)$ , are commutative, that is

$$(0.3) \quad \frac{\partial}{\partial x^b} \left( \frac{\partial f}{\partial x^a} \right) = \frac{\partial}{\partial x^a} \left( \frac{\partial f}{\partial x^b} \right)$$

The covariant derivative, however, is not in general commutative, as we can verify by direct calculation. We want to find

$$(0.4) \quad X^a_{b;c;d} - X^a_{b;d;c}$$

which is known as the commutator of the tensor  $X^a_b$ . For the first term, we get, using 0.2

$$(0.5) \quad X^a_{b;c;d} = \partial_d X^a_{b;c} + X^e_{b;c} \Gamma_{ed}^a - X^a_{e;c} \Gamma_{bd}^e - X^a_{b:e} \Gamma_{cd}^e$$

$$(0.6) \quad = \partial_d (\partial_c X^a_b + X^e_b \Gamma_{ec}^a - X^a_e \Gamma_{bc}^e) +$$

$$(0.7) \quad \Gamma_{ed}^a \left( \partial_c X^e_b + X^f_b \Gamma_{fc}^e - X^e_f \Gamma_{bc}^f \right) -$$

$$(0.8) \quad \Gamma_{bd}^e \left( \partial_c X^a_e + X^f_e \Gamma_{fc}^a - X^a_f \Gamma_{ec}^f \right) -$$

$$(0.9) \quad \Gamma_{cd}^e \left( \partial_e X^a_b + X^f_b \Gamma_{fe}^a - X^a_f \Gamma_{be}^f \right)$$

The other term can be obtained by simply swapping the indices  $c$  and  $d$ :

$$(0.10) \quad X^a_{b;d;c} = \partial_c X^a_{b;d} + X^e_{b;d} \Gamma^a_{ec} - X^a_{e;d} \Gamma^e_{bc} - X^a_{b;e} \Gamma^e_{dc}$$

$$(0.11) \quad = \partial_c (\partial_d X^a_b + X^e_b \Gamma^a_{ed} - X^a_e \Gamma^e_{bd}) +$$

$$(0.12) \quad \Gamma^a_{ec} (\partial_d X^e_b + X^f_b \Gamma^e_{fd} - X^e_f \Gamma^f_{bd}) -$$

$$(0.13) \quad \Gamma^e_{bc} (\partial_d X^a_e + X^f_e \Gamma^a_{fd} - X^a_f \Gamma^f_{ed}) -$$

$$(0.14) \quad \Gamma^e_{dc} (\partial_e X^a_b + X^f_b \Gamma^a_{fe} - X^a_f \Gamma^f_{be})$$

Now we need to take the difference. Assuming the ordinary partial derivatives commute and using the product rule, we get

$$(0.15) \quad X^a_{b;c;d} - X^a_{b;d;c} = X^e_b (\partial_d \Gamma^a_{ec} - \partial_c \Gamma^a_{ed}) - X^a_e (\partial_d \Gamma^e_{bc} - \partial_c \Gamma^e_{bd}) +$$

$$(0.16) \quad X^f_b (\Gamma^a_{ed} \Gamma^e_{fc} - \Gamma^a_{ec} \Gamma^e_{fd}) - X^e_f (\Gamma^a_{ed} \Gamma^f_{bc} - \Gamma^a_{ec} \Gamma^f_{bd}) -$$

$$(0.17) \quad X^f_e (\Gamma^e_{bd} \Gamma^a_{fc} - \Gamma^e_{bc} \Gamma^a_{fd}) + X^a_f (\Gamma^e_{bd} \Gamma^f_{ec} - \Gamma^e_{bc} \Gamma^f_{ed}) -$$

$$(0.18) \quad (\partial_e X^a_b + X^f_b \Gamma^a_{fe} - X^a_f \Gamma^f_{be}) (\Gamma^e_{cd} - \Gamma^e_{dc})$$

We can now swap the indices  $e$  and  $f$  in the first term in the third line (since they are both dummy indices) to get

$$(0.19) \quad X^f_e (\Gamma^e_{bd} \Gamma^a_{fc} - \Gamma^e_{bc} \Gamma^a_{fd}) = X^e_f (\Gamma^f_{bd} \Gamma^a_{ec} - \Gamma^f_{bc} \Gamma^a_{ed})$$

We can now see that this term cancels the last term on the second line. If we also assume that the affine connections are symmetric, so that

$$(0.20) \quad \Gamma^e_{cd} = \Gamma^e_{dc}$$

then the last line disappears and we are left with

$$(0.21) \quad X^a_{b;c;d} - X^a_{b;d;c} = X^e_b (\partial_d \Gamma^a_{ec} - \partial_c \Gamma^a_{ed}) - X^a_e (\partial_d \Gamma^e_{bc} - \partial_c \Gamma^e_{bd}) +$$

$$(0.22) \quad X^e_b (\Gamma^a_{fd} \Gamma^f_{ec} - \Gamma^a_{fc} \Gamma^f_{ed}) - X^a_e (\Gamma^f_{bc} \Gamma^e_{fd} - \Gamma^f_{bd} \Gamma^e_{fc})$$

$$(0.23) \quad = X^e_b (\partial_d \Gamma^a_{ec} - \partial_c \Gamma^a_{ed} + \Gamma^a_{fd} \Gamma^f_{ec} - \Gamma^a_{fc} \Gamma^f_{ed}) -$$

$$(0.24) \quad X^a_e (\partial_d \Gamma^e_{bc} - \partial_c \Gamma^e_{bd} + \Gamma^f_{bc} \Gamma^e_{fd} - \Gamma^f_{bd} \Gamma^e_{fc})$$

where again we have swapped  $e$  and  $f$  in the second line.

The two terms in parentheses have the same form, and they are known as the *Riemann tensor* or *curvature tensor*, defined by

$$(0.25) \quad R^a{}_{edc} \equiv \partial_d \Gamma^a{}_{ec} - \partial_c \Gamma^a{}_{ed} + \Gamma^a{}_{fd} \Gamma^f{}_{ec} - \Gamma^a{}_{fc} \Gamma^f{}_{ed}$$

In terms of the Riemann tensor, we get for the commutator:

$$(0.26) \quad X^a{}_{b;c;d} - X^a{}_{b;d;c} = X^e{}_b R^a{}_{edc} - X^a{}_e R^e{}_{bdc}$$

This is actually the same result as given in d'Inverno's problem 6.10, with  $c$  and  $d$  swapped around; I just took the original covariant derivatives in the opposite order to d'Inverno and can't be bothered going through the whole derivation again to change it.

#### PINGBACKS

Pingback: Riemann tensor and covariant contraction