

RIEMANN TENSOR - COMMUTATOR OF RANK 2 TENSOR

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Reference: d'Inverno, Ray, *Introducing Einstein's Relativity* (1992), Oxford Uni Press. - Section 6.5; Problem 6.10.

The covariant derivative of a contravariant vector is defined as

$$\nabla_b V^a \equiv V^a_{;b} \equiv \frac{\partial V^a}{\partial x^b} + V^c \Gamma_{cb}^a \quad (1)$$

This is generalized to the covariant derivative of a higher-rank tensor by the formula

$$T_{cd...;e}^{ab...} = \partial_e T_{cd...}^{ab...} + T_{cd...}^{fb...} \Gamma_{fe}^a + T_{cd...}^{af...} \Gamma_{fe}^b + \dots - T_{fd...}^{ab...} \Gamma_{ce}^f - T_{cf...}^{ab...} \Gamma_{de}^f - \dots \quad (2)$$

Ordinary partial derivatives, for a continuously differentiable function $f(x^a)$, are commutative, that is

$$\frac{\partial}{\partial x^b} \left(\frac{\partial f}{\partial x^a} \right) = \frac{\partial}{\partial x^a} \left(\frac{\partial f}{\partial x^b} \right) \quad (3)$$

The covariant derivative, however, is not in general commutative, as we can verify by direct calculation. We want to find

$$X^a_{b;c;d} - X^a_{b;d;c} \quad (4)$$

which is known as the commutator of the tensor X^a_b . For the first term, we get, using 2

$$X^a_{b;c;d} = \partial_d X^a_{b;c} + X^e_{b;c} \Gamma_{ed}^a - X^a_{e;c} \Gamma_{bd}^e - X^a_{b:e} \Gamma_{cd}^e \quad (5)$$

$$= \partial_d (\partial_c X^a_b + X^e_b \Gamma_{ec}^a - X^a_e \Gamma_{bc}^e) + \quad (6)$$

$$\Gamma_{ed}^a \left(\partial_c X^e_b + X^f_b \Gamma_{fc}^e - X^e_f \Gamma_{bc}^f \right) - \quad (7)$$

$$\Gamma_{bd}^e \left(\partial_c X^a_e + X^f_e \Gamma_{fc}^a - X^a_f \Gamma_{ec}^f \right) - \quad (8)$$

$$\Gamma_{cd}^e \left(\partial_e X^a_b + X^f_b \Gamma_{fe}^a - X^a_f \Gamma_{be}^f \right) \quad (9)$$

The other term can be obtained by simply swapping the indices c and d :

$$X^a_{b;d;c} = \partial_c X^a_{b;d} + X^e_{b;d} \Gamma^a_{ec} - X^a_{e;d} \Gamma^e_{bc} - X^a_{b:e} \Gamma^e_{dc} \quad (10)$$

$$= \partial_c (\partial_d X^a_b + X^e_b \Gamma^a_{ed} - X^a_e \Gamma^e_{bd}) + \quad (11)$$

$$\Gamma^a_{ec} (\partial_d X^e_b + X^f_b \Gamma^e_{fd} - X^e_f \Gamma^f_{bd}) - \quad (12)$$

$$\Gamma^e_{bc} (\partial_d X^a_e + X^f_e \Gamma^a_{fd} - X^a_f \Gamma^f_{ed}) - \quad (13)$$

$$\Gamma^e_{dc} (\partial_e X^a_b + X^f_b \Gamma^a_{fe} - X^a_f \Gamma^f_{be}) \quad (14)$$

Now we need to take the difference. Assuming the ordinary partial derivatives commute and using the product rule, we get

$$X^a_{b;c;d} - X^a_{b;d;c} = X^e_b (\partial_d \Gamma^a_{ec} - \partial_c \Gamma^a_{ed}) - X^a_e (\partial_d \Gamma^e_{bc} - \partial_c \Gamma^e_{bd}) + \quad (15)$$

$$X^f_b (\Gamma^a_{ed} \Gamma^e_{fc} - \Gamma^a_{ec} \Gamma^e_{fd}) - X^e_f (\Gamma^a_{ed} \Gamma^f_{bc} - \Gamma^a_{ec} \Gamma^f_{bd}) - \quad (16)$$

$$X^f_e (\Gamma^e_{bd} \Gamma^a_{fc} - \Gamma^e_{bc} \Gamma^a_{fd}) + X^a_f (\Gamma^e_{bd} \Gamma^f_{ec} - \Gamma^e_{bc} \Gamma^f_{ed}) - \quad (17)$$

$$(\partial_e X^a_b + X^f_b \Gamma^a_{fe} - X^a_f \Gamma^f_{be}) (\Gamma^e_{cd} - \Gamma^e_{dc}) \quad (18)$$

We can now swap the indices e and f in the first term in the third line (since they are both dummy indices) to get

$$X^f_e (\Gamma^e_{bd} \Gamma^a_{fc} - \Gamma^e_{bc} \Gamma^a_{fd}) = X^e_f (\Gamma^f_{bd} \Gamma^a_{ec} - \Gamma^f_{bc} \Gamma^a_{ed}) \quad (19)$$

We can now see that this term cancels the last term on the second line. If we also assume that the affine connections are symmetric, so that

$$\Gamma^e_{cd} = \Gamma^e_{dc} \quad (20)$$

then the last line disappears and we are left with

$$X^a_{b;c;d} - X^a_{b;d;c} = X^e_b (\partial_d \Gamma^a_{ec} - \partial_c \Gamma^a_{ed}) - X^a_e (\partial_d \Gamma^e_{bc} - \partial_c \Gamma^e_{bd}) + \quad (21)$$

$$X^e_b (\Gamma^a_{fd} \Gamma^f_{ec} - \Gamma^a_{fc} \Gamma^f_{ed}) - X^a_e (\Gamma^f_{bc} \Gamma^e_{fd} - \Gamma^f_{bd} \Gamma^e_{fc}) \quad (22)$$

$$= X^e_b (\partial_d \Gamma^a_{ec} - \partial_c \Gamma^a_{ed} + \Gamma^a_{fd} \Gamma^f_{ec} - \Gamma^a_{fc} \Gamma^f_{ed}) - \quad (23)$$

$$X^a_e (\partial_d \Gamma^e_{bc} - \partial_c \Gamma^e_{bd} + \Gamma^f_{bc} \Gamma^e_{fd} - \Gamma^f_{bd} \Gamma^e_{fc}) \quad (24)$$

where again we have swapped e and f in the second line.

The two terms in parentheses have the same form, and they are known as the *Riemann tensor* or *curvature tensor*, defined by

$$R^a{}_{edc} \equiv \partial_d \Gamma^a{}_{ec} - \partial_c \Gamma^a{}_{ed} + \Gamma^a{}_{fd} \Gamma^f{}_{ec} - \Gamma^a{}_{fc} \Gamma^f{}_{ed} \quad (25)$$

In terms of the Riemann tensor, we get for the commutator:

$$X^a{}_{b;c;d} - X^a{}_{b;d;c} = X^e{}_b R^a{}_{edc} - X^a{}_e R^e{}_{bdc} \quad (26)$$

This is actually the same result as given in d'Inverno's problem 6.10, with c and d swapped around; I just took the original covariant derivatives in the opposite order to d'Inverno and can't be bothered going through the whole derivation again to change it.

PINGBACKS

Pingback: Riemann tensor and covariant contraction