RIEMANN TENSOR AND COVARIANT CONTRACTION

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Reference: d'Inverno, Ray, *Introducing Einstein's Relativity* (1992), Oxford Uni Press. - Section 6.5; Problem 6.11.

Another exercise in fiddling with tensors.

The action of contracting a vector X with the covariant derivative of a tensor (not sure if this operation has an official name, but I'll call it *covariant contraction*) is (using the grad notation for covariant derivatives)

$$\nabla_X T^{ab\ldots}_{cd\ldots} \equiv X^e \nabla_e T^{ab\ldots}_{cd\ldots} \tag{0.1}$$

In the case of covariant contraction of a contravariant vector, we have

$$\nabla_X Z^a = X^e \nabla_e Z^a \tag{0.2}$$

If we combine two covariant contractions, we get

$$\begin{aligned}
\nabla_X \nabla_Y Z^a &= X^e \nabla_e \left(Y^c \nabla_c Z^a\right) && (0.3)\\
&= X^e \nabla_e Y^c \nabla_c Z^a + X^e Y^c \nabla_e \nabla_c Z^a && (0.4)
\end{aligned}$$

Swapping X and Y we get

$$\nabla_Y \nabla_X Z^a = Y^e \nabla_e X^c \nabla_c Z^a + Y^e X^c \nabla_e \nabla_c Z^a \tag{0.5}$$

We can swap the dummy indices c and e in the second term of this last equation to get

$$\nabla_Y \nabla_X Z^a = Y^e \nabla_e X^c \nabla_c Z^a + Y^c X^e \nabla_c \nabla_e Z^a \tag{0.6}$$

Taking the difference of these two derivatives we get

$$\begin{aligned}
\nabla_X \nabla_Y Z^a - \nabla_Y \nabla_X Z^a &= \left(X^e \nabla_e Y^c - Y^e \nabla_e X^c\right) \nabla_c Z^a + && (0.7)\\
& \quad X^e Y^c \left(\nabla_e \nabla_c Z^a - \nabla_c \nabla_e Z^a\right) && (0.8)
\end{aligned}$$

By following through a similar derivation to that given for a rank-2 tensor we find that the factor in the last term can be written in terms of the Riemann tensor:

$$\nabla_e \nabla_c Z^a - \nabla_c \nabla_e Z^a = R^a{}_{bec} Z^b \tag{0.9}$$

Also, the factor in the first term is a Lie bracket

$$X^e \nabla_e Y^c - Y^e \nabla_e X^c = [X, Y] \tag{0.10}$$

so we can write the first term as the covariant contraction with the vector given by the Lie bracket, or

$$\left(X^e \nabla_e Y^c - Y^e \nabla_e X^c\right) \nabla_c Z^a = \nabla_{[X,Y]} Z^a \tag{0.11}$$

Combining all this we get the final result

$$\nabla_X \nabla_Y Z^a - \nabla_Y \nabla_X Z^a - \nabla_{[X,Y]} Z^a = R^a{}_{bec} X^e Y^c Z^b \tag{0.12}$$