

RIEMANN TENSOR AND COVARIANT CONTRACTION

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Reference: d'Inverno, Ray, *Introducing Einstein's Relativity* (1992), Oxford Uni Press. - Section 6.5; Problem 6.11.

Another exercise in fiddling with tensors.

The action of contracting a vector X with the covariant derivative of a tensor (not sure if this operation has an official name, but I'll call it *covariant contraction*) is (using the grad notation for covariant derivatives)

$$(1) \quad \nabla_X T_{cd\dots}^{ab\dots} \equiv X^e \nabla_e T_{cd\dots}^{ab\dots}$$

In the case of covariant contraction of a contravariant vector, we have

$$(2) \quad \nabla_X Z^a = X^e \nabla_e Z^a$$

If we combine two covariant contractions, we get

$$(3) \quad \nabla_X \nabla_Y Z^a = X^e \nabla_e (Y^c \nabla_c Z^a)$$

$$(4) \quad = X^e \nabla_e Y^c \nabla_c Z^a + X^e Y^c \nabla_e \nabla_c Z^a$$

Swapping X and Y we get

$$(5) \quad \nabla_Y \nabla_X Z^a = Y^e \nabla_e X^c \nabla_c Z^a + Y^e X^c \nabla_e \nabla_c Z^a$$

We can swap the dummy indices c and e in the second term of this last equation to get

$$(6) \quad \nabla_Y \nabla_X Z^a = Y^e \nabla_e X^c \nabla_c Z^a + Y^c X^e \nabla_c \nabla_e Z^a$$

Taking the difference of these two derivatives we get

$$(7) \quad \nabla_X \nabla_Y Z^a - \nabla_Y \nabla_X Z^a = (X^e \nabla_e Y^c - Y^e \nabla_e X^c) \nabla_c Z^a +$$

$$(8) \quad X^e Y^c (\nabla_e \nabla_c Z^a - \nabla_c \nabla_e Z^a)$$

By following through a similar derivation to that given for a rank-2 tensor we find that the factor in the last term can be written in terms of the Riemann tensor:

$$(9) \quad \nabla_e \nabla_c Z^a - \nabla_c \nabla_e Z^a = R^a{}_{bec} Z^b$$

Also, the factor in the first term is a Lie bracket

$$(10) \quad X^e \nabla_e Y^c - Y^e \nabla_e X^c = [X, Y]$$

so we can write the first term as the covariant contraction with the vector given by the Lie bracket, or

$$(11) \quad (X^e \nabla_e Y^c - Y^e \nabla_e X^c) \nabla_c Z^a = \nabla_{[X, Y]} Z^a$$

Combining all this we get the final result

$$(12) \quad \nabla_X \nabla_Y Z^a - \nabla_Y \nabla_X Z^a - \nabla_{[X, Y]} Z^a = R^a{}_{bec} X^e Y^c Z^b$$