

## TANGENT SPACE: PARTIAL DERIVATIVES AS BASIS VECTORS

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Reference: d'Inverno, Ray, *Introducing Einstein's Relativity* (1992), Oxford Uni Press. - Section 5.9.

In our examination of tangent space, we came across the operator defined as

$$(1) \quad X \equiv \frac{dx^i}{dt} \frac{\partial}{\partial x^i}$$

Here it is assumed that we have a manifold with a coordinate system given by the  $x^i$ s, and that we have a curve embedded in this manifold given by the parametric equations

$$(2) \quad x^i = x^i(t)$$

A rather mysterious claim given in many books is that, if we apply these equations to a specific point in the manifold, we can take the operators  $\partial/\partial x^i$  as a basis for a vector space at that point. What on earth does that mean? Partial derivatives certainly don't look like vector components, and if your exposure to linear algebra doesn't go beyond the standard treatment of vectors, this claim can be baffling.

The idea is based on the observation that when we apply the operator  $X$  to a function  $f$ , we get the expansion for the total derivative of  $f$  w.r.t. the parameter  $t$ , that is, we get a directional derivative of  $f$  along the curve parametrized by  $t$ . This expansion, obtained by using the chain rule from calculus, is

$$(3) \quad \frac{df}{dt} = \frac{dx^i}{dt} \frac{\partial f}{\partial x^i}$$

If we regard  $\partial/\partial x^i$  as basis vectors, then the derivatives  $\frac{dx^i}{dt}$  are the components, and the total derivative  $\frac{df}{dt}$  is the vector formed as a result.

To see what this means, suppose we have a function of two variables, given as

$$(4) \quad f(x, y) = x^2 + y^2$$

This is a parabolic bowl with its base at the origin.

Now suppose we define the parametric curve

$$(5) \quad x(t) = t$$

$$(6) \quad y(t) = 2t$$

In more familiar terms, this is just the straight line  $y = 2x$ . The directional derivative of  $f$  as we move along this curve is therefore

$$(7) \quad \frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$$

Regarding the partial derivatives as basis vectors, the components of  $\frac{df}{dt}$  are then given by

$$(8) \quad \frac{dx}{dt} = 1$$

$$(9) \quad \frac{dy}{dt} = 2$$

These are the  $x$  and  $y$  components of the tangent vector to the surface defined by  $f$ , in the direction given by the parametric curve. The  $z$  component of this tangent vector can be found by recognizing that  $f$  is just  $z$ , so

$$(10) \quad \frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$$

$$(11) \quad = 2x + 4y$$

We can pick a specific point by choosing a value for  $t$ , say  $t = 2$ . Then  $x(2) = 2$  and  $y(2) = 4$  and

$$(12) \quad \left. \frac{dz}{dt} \right|_{t=2} = 20$$

$$(13) \quad z = x^2 + y^2$$

$$(14) \quad = 2^2 + 4^2$$

$$(15) \quad = 20$$

Thus the tangent vector to the surface  $f$  above the point  $p_0 = (2, 4, 20)$  is  $\mathbf{v}_1 = [1, 2, 20]$ .

As a check, we can verify that  $v$  is perpendicular to the normal line at  $p_0$ . The normal line to a surface (see any calculus textbook) is given in parametric form by

$$(16) \quad x(u) = p_{0x} + \left. \frac{\partial f}{\partial x} \right|_{p_0} u$$

$$(17) \quad y(u) = p_{0y} + \left. \frac{\partial f}{\partial y} \right|_{p_0} u$$

$$(18) \quad z(u) = p_{0z} - u$$

where the partial derivatives are evaluated at the point  $p_0$  on the surface.

To find a vector parallel to the line, we can choose two values of  $u$  and find the vector between the points. Choosing  $u = 0$  and  $u = 1$ , we get

$$(19) \quad x(0) = 2$$

$$(20) \quad y(0) = 4$$

$$(21) \quad z(0) = 20$$

$$(22) \quad x(1) = 6$$

$$(23) \quad y(1) = 12$$

$$(24) \quad z(1) = 19$$

The vector from the point  $u = 0$  to  $u = 1$  is therefore

$$(25) \quad \mathbf{n} = [4, 8, -1]$$

Taking the dot product, we get  $\mathbf{n} \cdot \mathbf{v}_1 = 4 + 16 - 20 = 0$  so the two vectors are orthogonal, as required.

The vector  $\mathbf{v}_1$  lies in the tangent plane to the surface  $f$  at the point  $p_0$ . The true meaning of taking the partial derivatives  $\partial/\partial x^i$  as basis vectors is that they are basis vectors for the tangent space (in this case, the tangent plane) to the function  $f$  at a particular point. The components of a given vector in this space are decided by choosing values for  $dx^i/dt$ , and we do this by choosing a particular parametric curve that goes through the point  $p_0$ , as we did above.

As another example, suppose we take the parametric curve

$$(26) \quad x(t) = t$$

$$(27) \quad y(t) = t^2$$

This is just the parabola  $y = x^2$ . If we take  $t = 2$ , we get the same values for  $x$  and  $y$  that we had in the last example, so the point on the surface defined by  $f$  is the same as before:  $p_0 = (2, 4, 20)$ . However, this time, the derivative  $dy/dt$  is different, so we have for the vector components

$$(28) \quad \frac{dx}{dt} = 1$$

$$(29) \quad \frac{dy}{dt} = 2t$$

$$(30) \quad \quad \quad = 4$$

The third component of the tangent vector is

$$(31) \quad \frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$$

$$(32) \quad \quad \quad = 2x(1) + 2y(4)$$

$$(33) \quad \quad \quad = 36$$

This time, the tangent vector is  $\mathbf{v}_2 = [1, 4, 36]$ . Since this is not a scalar multiple of the previous tangent vector  $\mathbf{v}_1$  it points in a different direction. We can verify that it is still in the tangent plane by calculating its dot product with the normal line as before:  $\mathbf{n} \cdot \mathbf{v}_2 = 4 + 32 - 36 = 0$ . Thus by choosing a different curve, we have generated a different vector in the tangent space.

In this process, note that the function  $f$  and the point  $p_0$  stay the same; it is the parametric curve that varies, and as a result the derivatives  $dx^i/dt$  vary. Because the function and point stay the same, the action of the operators  $\partial/\partial x^i$  also remains the same. That is why they can be regarded as the (fixed) basis vectors for the tangent space.

#### PINGBACKS

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