CONGRUENCE OF CURVES

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Reference: d'Inverno, Ray, *Introducing Einstein's Relativity* (1992), Oxford Uni Press. - Section 6.2.

Although tensor algebra is fairly straightforward, as tensors obey much the same rules as vectors, tensor calculus contains a few surprises. For example, the definition of the derivative of a tensor is not what you might expect, as just taking the derivative of a tensor directly does not give us an object that transforms the way a tensor should, and is therefore not itself a tensor.

There are, in fact, three ways of defining the derivative of a tensor. One of these, known as the *Lie derivative*, requires the concept of a *congruence of curves* within a manifold.

In discussing a congruence of curves, it's easiest to work in two dimensions, although of course the concept needs to be generalized to more dimensions in practice.

Within a plane there are various sets of curves we can define in such a way that only one curve within the set passes through any given point in the plane. For example, we might consider the set of all circles centred at the origin. None of these circles intersects any of the other circles, and for any point in the plane there is exactly one circle that contains it.

Since each point in the plane belongs to only one of the curves, we can define the tangent to that curve to be the tangent vector for that particular point. By this process, we can create a vector field, where the vector associated with each point is the tangent vector to the curve passing through that point.

We can write the equations of these curves in parametric form. For example, for the circles centred at the origin, we can write, for a circle of radius r

$$x(\theta) = r\cos\theta \tag{1}$$

$$y(\theta) = r\sin\theta \tag{2}$$

For a given circle, r is a constant, and the parameter θ varies from 0 to 2π. Choosing a different value of r gives us a different circle, but θ always varies over the same range.

In more general notation, we can write the congruence of curves as

$$x^a = x^a(u) \quad (3)$$

where u is the parameter. There will be other constants in these equations (such as r above) which determine which curve is being defined.

It is possible to reverse the process; that is, we can start with a vector field and derive the congruence from it. If u is the parameter defining the location on the curve, then the components of the tangent vector are defined by dx^a/du. If the components of the tangent vector field are written as X^a, then we have

$$\frac{dx^a}{du} = X^a(x(u)) \quad (4)$$

This is a set of (possibly coupled) differential equations that can be solved to give the congruence of curves as parametric equations.

To illustrate this, let's start with the simplest example. Suppose the vector field is constant, pointing in the x direction with a magnitude of 1 at every point. Then $X^a = (1,0)$ and the differential equations are (using standard $x-y$ notation rather than x^1, x^2):

$$\frac{dx}{du} = 1 \quad (5)$$

$$\frac{dy}{du} = 0 \quad (6)$$

These equations have the solutions

$$x = u + c_1 \quad (7)$$

$$y = c_2 \quad (8)$$

where c_1 and c_2 are constants whose values determine which curve we are considering.

The curves in this case are therefore straight lines parallel to the x axis. Since u varies from $-\infty$ to ∞, the value of c_1 is irrelevant, but c_2 determines how far from the x axis the straight line lies.

For a more involved example, suppose we have a vector field given by

$$X^a = [x+y, x-y] \quad (9)$$

Now the differential equations to be solved are

$$\frac{dx}{du} = x+y \tag{10}$$
$$\frac{dy}{du} = x-y \tag{11}$$

This system can be solved (using software, for example, or by standard ODE techniques) to give

$$x(u) = c_1 e^{\sqrt{2}u} + c_2 e^{-\sqrt{2}u} \tag{12}$$
$$y(u) = c_1\left(\sqrt{2}-1\right)e^{\sqrt{2}u} - c_2\left(\sqrt{2}+1\right)e^{-\sqrt{2}u} \tag{13}$$

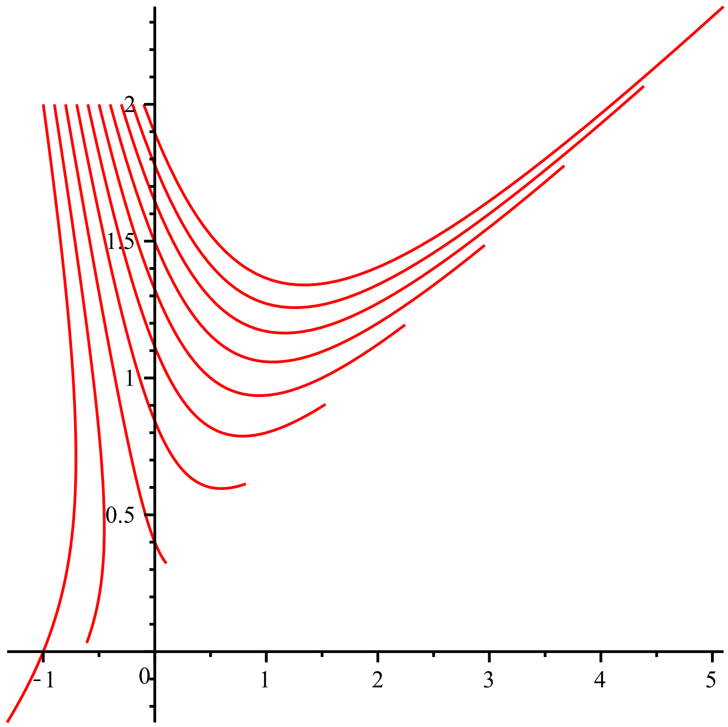
The constants c_1 and c_2 are determined by specifying the values of x and y for a given value of u. Typically this is done by specifying the initial conditions, that is $x(0) \equiv x_0$ and $y(0) \equiv y_0$. In this case,

$$c_1 + c_2 = x_0 \tag{14}$$
$$c_1\left(\sqrt{2}-1\right) - c_2\left(\sqrt{2}+1\right) = y_0 \tag{15}$$

which can be solved to give

$$c_1 = \frac{\sqrt{2}}{4}\left[\left(\sqrt{2}+1\right)x_0 + y_0\right] \tag{16}$$
$$c_2 = \frac{\sqrt{2}}{4}\left[\left(\sqrt{2}-1\right)x_0 - y_0\right] \tag{17}$$

The curves given by these parametric equations depend, of course, on the initial conditions, but here's a sample for the case $y_0 = 2$ and x_0 varying from -1 to -0.1 in steps of 0.1.



The point on each curve corresponding to $u=0$ can be found by drawing the horizontal line through $y=2$, so we see that the curves run from left to right (that is, the curve for $x_0=-1$ is on the left). In each case, u runs from 0 to 1.5.

We'd need to do a fair bit more work to show that (a) none of the curves for any value of c_1 and c_2 intersect and (b) the entire plane is covered by these curves, but we can at least see from the equations above that values of c_1 and c_2 exist for any values of x_0 and y_0.

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