LORENTZ TRANSFORMATIONS: DERIVATION FROM SYMMETRY

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Reference: Carroll, Bradley W. & Ostlie, Dale A. (2007), *An Introduction to Modern Astrophysics*, 2nd Edition; Pearson Education - Chapter 4, Problem 4.1.

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Although we've already looked at special relativity several times in this blog, it's worth working through Chapter 4 in Carroll & Ostlie since they offer a few different ways of looking at some of our previous results.

We can start with the Lorentz transformations. The derivation we studied most recently is that from Griffiths's book on electromagnetism, in which he first derives the time dilation and length contraction effects and then uses these to derive the Lorentz transformations. Carroll & Ostlie take a somewhat simpler and more elegant approach, but there are still a few points that could be filled in.

The arguments rely on using various symmetries, and also the postulate of the constancy of the speed of light to finish things off.

First, we can use translational invariance to show that the Lorentz transformations must be linear. We've already shown that this is the case using a rather involved argument, but in fact there is a simple criterion that can be applied. We start by using the line-painting thought experiment to show that coordinates perpendicular to the direction of relative motion are unaffected, so if we use our usual two coordinate systems S and S', with S at rest relative to the observer, S' moving with speed u in the $+x$ direction and the two frames aligned so that all three of their coordinate axis pairs (x and x', y and y' and z and z') are parallel with the origins coinciding at $t=t'=0$, then

$$y' = y \tag{1}$$

$$z' = z \tag{2}$$

For the remaining two coordinates, the most general linear transformation is

$$x' = a_{11}x + a_{12}y + a_{13}z + a_{14}t \quad (3)$$
$$t' = a_{41}x + a_{42}y + a_{43}z + a_{44}t \quad (4)$$

Why linear? Well, suppose we consider the length of a rod in the two frames. The rod is at rest in S with one endpoint at $x_1 = 0$ and the other at $x_2 = L$. We know that S and S' disagree about the length of the rod, but one thing we are sure of is that each observer will obtain only one result for the length. In S the length is L and in S' the length is L'. But suppose we changed the origin in S and S' by shifting it along the x axis by a distance of 1. Then in S, where the rod is at rest, the coordinates of its endpoints are now $x_1 = -1$ and $x_2 = L-1$ so that the length is still given by $x_2 - x_1 = L$. Now whatever the Lorentz transformation is, it has to give L' for the length as measured by S'. In the original frames (before we shifted the origins) the length in S' (taking the rod to lie on the x axis so that $y = z = 0$)

$$L' = x_2' - x_1' \quad (5)$$
$$= a_{11}(x_2 - x_1) + a_{14}(t_2 - t_1) \quad (6)$$
$$= a_{11}(x_2 - x_1) \quad (7)$$
$$= a_{11}(L - 0) \quad (8)$$
$$= a_{11}L \quad (9)$$

where the third line is true because the two events defining the measurement of the length of the rod occur at the same time in S so $t_1 = t_2$. Now if we use the shifted origins

$$L' = a_{11}(x_2 - x_1) + a_{14}(t_2 - t_1) \quad (10)$$
$$= a_{11}(L - 1 - (-1)) \quad (11)$$
$$= a_{11}L \quad (12)$$

Thus shifting the origin leaves the length L' the same. However, if we use a *non*-linear transformation $f(x)$ instead of the $a_{11}x$ term, then with the original origins

$$L' = f(L) - f(0) \quad (13)$$

and with the shifted origins

$$L' = f(L-1) - f(-1) \quad (14)$$

and in general these two values won't be equal. Even if we do find some non-linear function that gives the same values for these particular choices for x_1 and x_2, what we really need is a transformation that gives the same values for L' for *all* lengths, at any location on the x axis. The only transformation that does that is linear.

So much for translational symmetry. Next, we can apply rotational symmetry. If we rotate both coordinates systems by 180° about the x axis so that y goes to $-y$ and z to $-z$, all we've done is change the coordinate system used to describe the problem; we haven't actually changed any of the events that occur. Thus, the equations 3 and 4 must give the same results with $y \to -y$ and $z \to -z$. By choosing an event with $y \neq 0$ and $z = 0$ we have

$$a_{11}x + a_{12}y + a_{14}t = a_{11}x - a_{12}y + a_{14}t \tag{15}$$
$$a_{12}y = -a_{12}y \tag{16}$$

so we conclude that $a_{12} = 0$. Choosing $y = 0$ and $z \neq 0$ gives us $a_{13} = 0$. Thus 3 becomes

$$x' = a_{11}x + a_{14}t \tag{17}$$

A similar argument applied to 4 gives $a_{42} = a_{43} = 0$ so

$$t' = a_{41}x + a_{44}t \tag{18}$$

The origin of S' is moving to the right with speed u in S, so at time t its x coordinate is $x = ut$, but $x' = 0$ always since the origin of S' is at rest in S'. Therefore 17 becomes for this origin

$$0 = a_{11}ut + a_{14}t \tag{19}$$
$$a_{14} = -a_{11}u \tag{20}$$

The transformations up to this point are

$$x' = a_{11}(x - ut) \tag{21}$$
$$y' = y \tag{22}$$
$$z' = z \tag{23}$$
$$t' = a_{41}x + a_{44}t \tag{24}$$

To get the final three constants, we need to invoke the constancy of the speed of light c. Suppose that at $t = t' = 0$ a pulse of light is generated at the common origins of S and S'. Because c is the same in both frames, both observers will see a spherical shell of light expand from their respective origins. The equations of this shell in the two frames are of the same form:

$$x^2+y^2+z^2 = (ct)^2 \tag{25}$$
$$x'^2+y'^2+z'^2 = \left(ct'\right)^2 \tag{26}$$

The last equation gives

$$a_{11}^2(x-ut)^2+y^2+z^2=c^2(a_{41}x+a_{44}t)^2 \tag{27}$$
$$x^2\left(a_{11}^2-a_{41}^2c^2\right)+y^2+z^2=t^2\left(c^2a_{44}^2-u^2a_{11}^2\right)+2xt\left(c^2a_{41}a_{44}+ua_{11}^2\right) \tag{28}$$

Comparing this with 25 we get

$$a_{11}^2-a_{41}^2c^2 = 1 \tag{29}$$
$$c^2a_{44}^2-u^2a_{11}^2 = c^2 \tag{30}$$
$$c^2a_{41}a_{44}+ua_{11}^2 = 0 \tag{31}$$

From 29, we have

$$a_{11}^2=1+c^2a_{41}^2 \tag{32}$$

Substitute this into 31:

$$c^2a_{41}a_{44}+u\left(1+c^2a_{41}^2\right)=0 \tag{33}$$

Multiply the first equation by u^2 and add to the second, and multiply the first equation by u and subtract the third:

$$c^2a_{44}^2-a_{41}^2u^2c^2 = c^2+u^2 \tag{34}$$
$$-a_{41}^2c^2u-c^2a_{41}a_{44} = u \tag{35}$$

Multiply the second equation by $-u$ and add to the first:

$$c^2a_{44}^2+uc^2a_{44}a_{41}-c^2 = 0 \tag{36}$$
$$a_{44}^2+ua_{44}a_{41}-1 = 0 \tag{37}$$

Using 33:

$$a_{44}=-\frac{u\left(a_{41}^2c^2+1\right)}{c^2a_{41}} \tag{38}$$

If we plug this into 37 and multiply through by a_{41}^2, we get

$$-\left(1-\frac{u^2}{c^2}\right)a_{41}^2+\frac{u^2}{c^4}=0 \tag{39}$$

so

$$a_{41}=\pm\frac{u}{c^2\sqrt{1-u^2/c^2}} \tag{40}$$

Taking the negative root and plugging this back into 38 gives

$$a_{44}=\frac{\sqrt{c^2-u^2}}{c}\left(\frac{u^2}{c^2-u^2}+1\right) \tag{41}$$

$$=\frac{1}{\sqrt{1-u^2/c^2}} \tag{42}$$

From 30:

$$a_{11}^2 = \frac{c^2}{u^2}\left(a_{44}^2-1\right) \tag{43}$$

$$= \frac{c^2}{u^2}\left(\frac{u^2/c^2}{1-u^2/c^2}\right) \tag{44}$$

$$= \frac{1}{1-u^2/c^2} \tag{45}$$

$$a_{11} = \frac{1}{\sqrt{1-u^2/c^2}} \tag{46}$$

Putting it all together gives the familiar Lorentz transformations:

$$x' = \frac{1}{\sqrt{1-u^2/c^2}}(x-ut) \tag{47}$$

$$y' = y \tag{48}$$

$$z' = z \tag{49}$$

$$t' = \frac{t-ux/c^2}{\sqrt{1-u^2/c^2}} \tag{50}$$

PINGBACKS

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