

LAGRANGIAN FOR THE TWO-BODY PROBLEM

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A fundamental problem in classical physics is the two-body problem, in which two masses interact via a potential $V(\mathbf{r}_1 - \mathbf{r}_2)$ that depends only on the relative positions of the two masses. In such a case, the Lagrangian can be decoupled so that the problem gets reduced to a one-body problem.

The Euler-Lagrange equations are

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \quad (1)$$

where q_i and \dot{q}_i are the generalized coordinates and velocities, respectively. For systems where the potential energy $V(q_i)$ is independent of the velocities \dot{q}_i , the Lagrangian can be written as

$$L = T - V \quad (2)$$

where T is the kinetic energy. In terms of the absolute positions and velocities, we have

$$L = \frac{1}{2} m_1 |\dot{\mathbf{r}}_1|^2 + \frac{1}{2} m_2 |\dot{\mathbf{r}}_2|^2 - V(\mathbf{r}_1 - \mathbf{r}_2) \quad (3)$$

To decouple this equation, we define two new position vectors:

$$\begin{aligned} \mathbf{r} &\equiv \mathbf{r}_1 - \mathbf{r}_2 \\ \mathbf{r}_{CM} &\equiv \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2} \end{aligned} \quad (4)$$

Here \mathbf{r} is the relative position, and \mathbf{r}_{CM} is the position of the centre of mass.

We can invert these equations to get

$$\mathbf{r}_1 = \mathbf{r} + \mathbf{r}_2 \quad (5)$$

$$(m_1 + m_2) \mathbf{r}_{CM} = m_1 \mathbf{r} + (m_1 + m_2) \mathbf{r}_2 \quad (6)$$

$$\mathbf{r}_2 = \mathbf{r}_{CM} - \frac{m_1}{m_1 + m_2} \mathbf{r} \quad (7)$$

$$\mathbf{r}_1 = \mathbf{r}_{CM} - \frac{m_2}{m_1 + m_2} \mathbf{r} \quad (8)$$

To decouple the Lagrangian, we insert these last two equations into 3.

$$\begin{aligned}
 m_1 |\dot{\mathbf{r}}_1|^2 &= m_1 \left[\dot{\mathbf{r}}_{CM} - \frac{m_2}{m_1 + m_2} \dot{\mathbf{r}} \right] \cdot \left[\dot{\mathbf{r}}_{CM} - \frac{m_2}{m_1 + m_2} \dot{\mathbf{r}} \right] \\
 &= m_1 |\dot{\mathbf{r}}_{CM}|^2 - 2 \frac{m_1 m_2}{m_1 + m_2} \dot{\mathbf{r}}_{CM} \cdot \dot{\mathbf{r}} + m_1 \left(\frac{m_2}{m_1 + m_2} \right)^2 |\dot{\mathbf{r}}|^2
 \end{aligned} \tag{9}$$

$$\begin{aligned}
 m_2 |\dot{\mathbf{r}}_2|^2 &= m_2 \left[\dot{\mathbf{r}}_{CM} + \frac{m_1}{m_1 + m_2} \dot{\mathbf{r}} \right] \cdot \left[\dot{\mathbf{r}}_{CM} + \frac{m_1}{m_1 + m_2} \dot{\mathbf{r}} \right] \\
 &= m_2 |\dot{\mathbf{r}}_{CM}|^2 + 2 \frac{m_1 m_2}{m_1 + m_2} \dot{\mathbf{r}}_{CM} \cdot \dot{\mathbf{r}} + m_2 \left(\frac{m_1}{m_1 + m_2} \right)^2 |\dot{\mathbf{r}}|^2
 \end{aligned} \tag{10}$$

$$\frac{1}{2} m_1 |\dot{\mathbf{r}}_1|^2 + \frac{1}{2} m_2 |\dot{\mathbf{r}}_2|^2 = \frac{1}{2} (m_1 + m_2) |\dot{\mathbf{r}}_{CM}|^2 + \frac{1}{2} \frac{m_1 m_2^2 + m_2 m_1^2}{(m_1 + m_2)^2} |\dot{\mathbf{r}}|^2 \tag{11}$$

$$= \frac{1}{2} (m_1 + m_2) |\dot{\mathbf{r}}_{CM}|^2 + \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} |\dot{\mathbf{r}}|^2 \tag{12}$$

The Lagrangian 3 thus becomes

$$L = \frac{1}{2} (m_1 + m_2) |\dot{\mathbf{r}}_{CM}|^2 + \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} |\dot{\mathbf{r}}|^2 - V(\mathbf{r}) \tag{13}$$

$$\equiv L_{CM} + L_r \tag{14}$$

with

$$L_{CM} \equiv \frac{1}{2} (m_1 + m_2) |\dot{\mathbf{r}}_{CM}|^2 \tag{15}$$

$$L_r \equiv \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} |\dot{\mathbf{r}}|^2 - V(\mathbf{r}) \tag{16}$$

Thus L decouples into two Lagrangians, one of which depends only on $\dot{\mathbf{r}}_{CM}$ and the other of which depends only on \mathbf{r} and $\dot{\mathbf{r}}$. The absence of \mathbf{r}_{CM} means that, from 1

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{r}_{i,CM}} = \frac{d}{dt} \frac{\partial L_{CM}}{\partial \dot{r}_{i,CM}} = \frac{m_1 + m_2}{2} \frac{d \dot{r}_{i,CM}}{dt} = 0 \tag{17}$$

$$\dot{r}_{i,CM} = \text{constant} \tag{18}$$

which is separately true for each component of $\dot{\mathbf{r}}_{CM}$, which shows that the velocity of the centre of mass is a constant, as we'd expect for an isolated two-body system with no external force.

From the other Lagrangian, we get

$$\frac{m_1 m_2}{m_1 + m_2} \ddot{\mathbf{r}} = -\nabla V(\mathbf{r}) \quad (21)$$

which is the equation of motion of a single particle of mass $\frac{m_1 m_2}{m_1 + m_2}$, called the *reduced mass*. Viewed from the centre of mass frame, where $\dot{\mathbf{r}}_{CM} = 0$, \mathbf{r} becomes the absolute position of the reduced mass. We can transform the result back to the 'absolute' frame by using 4.

PINGBACKS

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