RELATIVISTIC INVARIANT INTEGRATION MEASURE

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Reference: Sidney Coleman, *Quantum Field Theory: Lectures of Sidney Coleman* (edited by Bryan Gin-ge Chen *et al.*), World Scientific, 2019. Section 1.2.

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We look at a simple calculation here that is important because it's used many times in doing integrals in quantum field theory. In many cases, we need to do an integral over energy-momentum space of the form

$$\int d^4p\, f(p) \tag{1}$$

where $f(p)$ is some function of the 4-momentum p.

First, we observe that the integration measure, d^4p, is Lorentz invariant. This is because, under a Lorentz transformation represented by the 4×4 matrix Λ, the 4-momentum transforms as

$$p \to \Lambda p \tag{2}$$

This is a linear transformation (the elements of Λ depend only on things like the relative speed of two inertial frames, or, for a rotation, the angle of rotation, but not on the components of the 4-momentum p). Therefore Λ is itself the Jacobian of the transformation. Thus the integration measure transforms according to

$$d^4p \to \left|\det\Lambda\right| d^4p \tag{3}$$

We've seen that

$$\det\Lambda = \pm 1 \tag{4}$$

so the measure d^4p remains invariant under a Lorentz transformation.

However, in many applications, we wish to restrict the integration over d^4p to the *mass shell*, that is, the set of 4-momenta that obey the energy-momentum relation

$$p^0 = \omega_{\boldsymbol{p}} = +\sqrt{|\boldsymbol{p}|^2 + \mu^2} \tag{5}$$

where $\boldsymbol{p}$ is the 3-momentum and μ is the mass of the particle. That is, the constraint becomes

$$p^2 = p_\nu p^\nu = \mu^2 \tag{6}$$

If we're given an integral of the form 1, we can enforce this constraint by using a delta function:

$$\int d^4p\, f(p) = \int dp^0 \left[d^3\boldsymbol{p}\delta\left(p^2-\mu^2\right)\theta\left(p^0\right)\right] f(p) \tag{7}$$

The step function $\theta\left(p^0\right)$ restricts p^0 to $p^0 > 0$.

If we consider only the integration over p^0, we have

$$\int_{-\infty}^{\infty} dp^0 \left[d^3\boldsymbol{p}\delta\left(p^2-\mu^2\right)\theta\left(p^0\right)\right] f(p) \tag{8}$$

We can transform the delta function using the identity

$$\delta\left(g(x)\right) = \sum_i \frac{\delta\left(x-z_i\right)}{\left|g'\left(z_i\right)\right|} \tag{9}$$

where z_i are the zeroes of $g(x)$. Note in particular that $g'(z_i)$ is the *derivative* of g evaluated at points where the original function $g(x)$ (*not* the derivative!) is zero.

In this case,

$$p^2-\mu^2 = \left(p^0\right)^2 - \boldsymbol{p}^2 - \mu^2 \tag{10}$$

$$= \left(p^0\right)^2 - \omega_{\boldsymbol{p}}^2 \tag{11}$$

so we have

$$g\left(p^0\right) = \left(p^0\right)^2 - \omega_{\boldsymbol{p}}^2 \tag{12}$$

$$g'\left(p^0\right) = 2p^0 \tag{13}$$

The function $g\left(p^0\right)$ has zeroes at $p^0 = \pm\omega_{\boldsymbol{p}}$, but since the step function $\theta\left(p^0\right)$ restricts p^0 to $p^0 > 0$, we need consider only the zero at $p^0 = \omega_{\boldsymbol{p}}$. At this point $g'\left(\omega_{\boldsymbol{p}}\right) = 2\omega_{\boldsymbol{p}}$, so the formula 9 gives us

$$\delta\left(p^2-\mu^2\right) = \frac{\delta\left(p^0-\omega_{\boldsymbol{p}}\right)}{2\omega_{\boldsymbol{p}}} \tag{14}$$

Inserting this into 8 gives us

$$\int_{-\infty}^{\infty} dp^0 \left[d^3\boldsymbol{p}\delta\left(p^2-\mu^2\right)\theta\left(p^0\right)\right] f(p) = \int_0^{\infty} dp^0 \frac{\delta\left(p^0-\omega_{\boldsymbol{p}}\right)}{2\omega_{\boldsymbol{p}}} d^3\boldsymbol{p} f(p) \quad (15)$$

$$= \frac{d^3\boldsymbol{p}}{2\omega_{\boldsymbol{p}}} f(\boldsymbol{p}) \quad (16)$$

where all occurrences of p^0 in the function $f(p)$ have been replaced by $\omega_{\boldsymbol{p}}$, so that, from 5, $f(p)$ becomes a function of the 3-momentum $f(\boldsymbol{p})$, since p^0 is now written entirely in terms of $\boldsymbol{p}$.

The Lorentz invariant measure d^4p on the mass shell is thus equivalent to the measure

$$d^4p \rightarrow \frac{d^3\boldsymbol{p}}{2\omega_{\boldsymbol{p}}} \quad (17)$$

and it is this measure (not $d^3\boldsymbol{p}$ on its own) that is Lorentz invariant.

This measure is used to define a normalization for a relativistic ket $|p\rangle$:

$$|p\rangle = \sqrt{(2\pi)^3}\sqrt{2\omega_{\boldsymbol{p}}}\,|\boldsymbol{p}\rangle \quad (18)$$

The usual unit operator

$$1 = \int d^3\boldsymbol{p}\ |\boldsymbol{p}\rangle\langle\boldsymbol{p}| \quad (19)$$

can therefore be written

$$1 = \frac{1}{(2\pi)^3}\int d^4p\ \delta\left(p^2-\mu^2\right)\theta\left(p^0\right)|p\rangle\langle p| \quad (20)$$

$$= \frac{1}{(2\pi)^3}\int \frac{d^3\boldsymbol{p}}{2\omega_{\boldsymbol{p}}}(2\pi)^3\, 2\omega_{\boldsymbol{p}}\,|\boldsymbol{p}\rangle\langle\boldsymbol{p}| \quad (21)$$

$$= \int d^3\boldsymbol{p}\ |\boldsymbol{p}\rangle\langle\boldsymbol{p}| \quad (22)$$

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