

## LORENTZ TRANSFORMATIONS OF RELATIVISTIC STATES

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Reference: Sidney Coleman, *Quantum Field Theory: Lectures of Sidney Coleman* (edited by Bryan Gin-ge Chen *et al.*), World Scientific, 2019. Section 1.2.

Post date: 9 Dec 2019.

The relativistic state of a single particle with 4-momentum  $p$  is defined as

$$|p\rangle = \sqrt{(2\pi)^3} \sqrt{2\omega_{\mathbf{p}}} |\mathbf{p}\rangle \quad (1)$$

That is, it's defined in terms of the state  $|\mathbf{p}\rangle$  with 3-momentum  $\mathbf{p}$  and energy

$$\omega_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + \mu^2} \quad (2)$$

We also showed earlier that integration over the Lorentz-invariant integration measure  $d^4p$  is equivalent to integration over the 3-momentum space with measure  $d^3\mathbf{p}/2\omega_{\mathbf{p}}$ . That is

$$\int d^4p f(p) = \int \frac{d^3\mathbf{p}}{2\omega_{\mathbf{p}}} f(\mathbf{p}) \quad (3)$$

where  $p^0 = \omega_{\mathbf{p}}$ , so that any function of the 4-momentum  $p$  becomes a function of the 3-momentum  $\mathbf{p}$ .

The unit operator can then be written as

$$1 = \frac{1}{(2\pi)^3} \int \frac{d^3\mathbf{p}}{2\omega_{\mathbf{p}}} |p\rangle \langle p| \quad (4)$$

Note that the states in this integral are 4-momentum states  $|p\rangle$  and not 3-momentum states  $|\mathbf{p}\rangle$ .

For a Lorentz transformation  $\Lambda$ , we can define a unitary operator  $U(\Lambda)$  which acts on a single particle ket as follows.

$$U(\Lambda) |p\rangle = |\Lambda p\rangle \quad (5)$$

This operator satisfies four properties which can be derived using methods similar to Coleman's equations 1.48 - 1.50, where he derives similar rules for the rotation operator.

First, we have

$$U(\Lambda)U^\dagger(\Lambda) = 1 \quad (6)$$

We can prove this as follows.

$$U(\Lambda)U^\dagger(\Lambda) = U(\Lambda) \frac{1}{(2\pi)^3} \int \frac{d^3\mathbf{p}}{2\omega_{\mathbf{p}}} |p\rangle \langle p| U^\dagger(\Lambda) \quad (7)$$

$$= \frac{1}{(2\pi)^3} \int \frac{d^3\mathbf{p}}{2\omega_{\mathbf{p}}} U(\Lambda) |p\rangle \langle p| U^\dagger(\Lambda) \quad (8)$$

$$= \frac{1}{(2\pi)^3} \int \frac{d^3\mathbf{p}}{2\omega_{\mathbf{p}}} |\Lambda p\rangle \langle \Lambda p| \quad (9)$$

As we showed earlier, transforming 4-momenta with a Lorentz transformation has a Jacobian of 1, so the integration measure  $d^4p$  is unchanged. Since integrating over this measure is equivalent to integrating over 3-momenta via 3, we can define  $p' \equiv \Lambda p$  and then we have

$$\frac{d^3\mathbf{p}'}{2\omega_{\mathbf{p}'}} = \frac{d^3\mathbf{p}}{2\omega_{\mathbf{p}}} \quad (10)$$

We therefore have

$$U(\Lambda)U^\dagger(\Lambda) = \frac{1}{(2\pi)^3} \int \frac{d^3\mathbf{p}'}{2\omega_{\mathbf{p}'}} |p'\rangle \langle p'| = 1 \quad (11)$$

The next property is

$$U(1) = 1 \quad (12)$$

which follows directly from 5, since if  $\Lambda = 1$ , it is the identity matrix which amounts to no transformation at all, which leaves the state unchanged.

Next, we have the property

$$U(\Lambda_1)U(\Lambda_2) = U(\Lambda_1\Lambda_2) \quad (13)$$

We can see this directly from 5:

$$U(\Lambda_1)U(\Lambda_2)|p\rangle = U(\Lambda_1)|\Lambda_2 p\rangle \quad (14)$$

$$= |\Lambda_1\Lambda_2 p\rangle \quad (15)$$

$$= U(\Lambda_1\Lambda_2)|p\rangle \quad (16)$$

Finally we have

$$U^\dagger(\Lambda)PU(\Lambda) = \Lambda P \quad (17)$$

where  $P$  is the 4-momentum operator.

Because  $U$  is unitary, we have  $U^\dagger = U^{-1}$  (we proved this when we proved 6) so

$$U^\dagger(\Lambda) P U(\Lambda) = U^{-1}(\Lambda) P \left( U^\dagger(\Lambda) \right)^{-1} \quad (18)$$

$$= U(\Lambda^{-1}) P U^\dagger(\Lambda^{-1}) \quad (19)$$

This follows because

$$U^{-1}(\Lambda) U(\Lambda) = 1 \quad (20)$$

and

$$U(\Lambda^{-1}) U(\Lambda) = U(\Lambda^{-1}\Lambda) = U(1) = 1 \quad (21)$$

Therefore

$$U(\Lambda^{-1}) = U^{-1}(\Lambda) \quad (22)$$

Continuing from 19 and using 4

$$U U^\dagger(\Lambda) P U(\Lambda) = U(\Lambda^{-1}) P \frac{1}{(2\pi)^3} \int \frac{d^3 \mathbf{p}}{2\omega_{\mathbf{p}}} |p\rangle \langle p| U^\dagger(\Lambda^{-1}) \quad (23)$$

$$= \frac{1}{(2\pi)^3} U(\Lambda^{-1}) \int \frac{d^3 \mathbf{p}}{2\omega_{\mathbf{p}}} p |p\rangle \langle p| U^\dagger(\Lambda^{-1}) \quad (24)$$

$$= \frac{1}{(2\pi)^3} \int \frac{d^3 \mathbf{p}}{2\omega_{\mathbf{p}}} p U(\Lambda^{-1}) |p\rangle \langle p| U^\dagger(\Lambda^{-1}) \quad (25)$$

$$= \frac{1}{(2\pi)^3} \int \frac{d^3 \mathbf{p}}{2\omega_{\mathbf{p}}} p |\Lambda^{-1} p\rangle \langle \Lambda^{-1} p| \quad (26)$$

We now let  $p = \Lambda p'$  and use the invariance of the integration measure  $\frac{d^3 \mathbf{p}'}{2\omega_{\mathbf{p}'}} = \frac{d^3 \mathbf{p}}{2\omega_{\mathbf{p}}}$ :

$$U^\dagger(\Lambda) P U(\Lambda) = \frac{1}{(2\pi)^3} \int \frac{d^3 \mathbf{p}'}{2\omega_{\mathbf{p}'}} \Lambda p' |p'\rangle \langle p'| \quad (27)$$

$$= \frac{1}{(2\pi)^3} \int \frac{d^3 \mathbf{p}}{2\omega_{\mathbf{p}}} \Lambda p |p\rangle \langle p| \quad (28)$$

$$= \Lambda P \left[ \frac{1}{(2\pi)^3} \int \frac{d^3 \mathbf{p}}{2\omega_{\mathbf{p}}} |p\rangle \langle p| \right] \quad (29)$$

$$= \Lambda P \quad (30)$$

where in the third line, the integral in the brackets is the unit operator from 4. QED.