FREE SPINLESS PARTICLE AND CAUSALITY

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Reference: Sidney Coleman, *Quantum Field Theory: Lectures of Sidney Coleman* (edited by Bryan Gin-ge Chen *et al.*), World Scientific, 2019. Section 1.3.

Phillipe Dennery & André Krzywicki, *Mathematics for Physicists*, Dover, 1996, pp 71-73.

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In section 1.3, Coleman works out the probability that the spinless particle governed by the equation

$$H|\boldsymbol{p}\rangle=\sqrt{|\boldsymbol{p}|^2+\mu^2}\,|\boldsymbol{p}\rangle \tag{1}$$

can violate causality in special relativity by travelling faster than light. If the particle starts at $\boldsymbol{x}=0$ at time $t=0$, then the probability of it being found at a general location $\boldsymbol{x}$ at some future time t is

$$\left\langle \boldsymbol{x}\left|e^{-iHt}\right|\psi\right\rangle \tag{2}$$

where $|\psi\rangle$ is the state of the particle at $t=0$.

Coleman shows that this probability works out to

$$\left\langle \boldsymbol{x}\left|e^{-iHt}\right|\psi\right\rangle=\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}e^{\boldsymbol{p}\cdot\boldsymbol{x}}e^{-i\omega_{\boldsymbol{p}}t} \tag{3}$$

where

$$\omega_{\boldsymbol{p}}=\sqrt{|\boldsymbol{p}|^2+\mu^2} \tag{4}$$

By going over to polar coordinates, Coleman shows that this integral comes out to

$$\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}e^{\boldsymbol{p}\cdot\boldsymbol{x}}e^{-i\omega_{\boldsymbol{p}}t}=-\frac{i}{(2\pi)^2r}\int_{-\infty}^{\infty}dp\,p\,e^{ipr-i\omega_p t} \tag{5}$$

where $r=|\boldsymbol{x}|$ and $p=|\boldsymbol{p}|$.

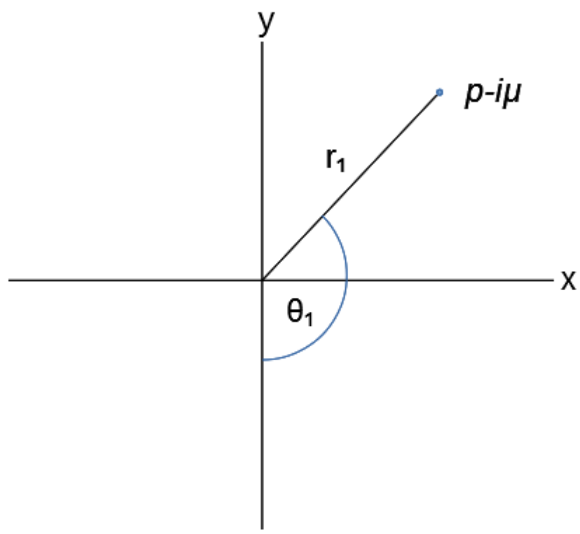
The integral can be done (well, partly done) by using contour integration, in which we consider p to be a complex number. The problem is that ω_p is a double-valued function, since the square root can be positive or negative.

To do the contour integral, we therefore need to consider the branch points and branch cuts of ω_p.

We can write

$$\omega_p = \sqrt{(p+i\mu)(p-i\mu)} \tag{6}$$

Consider the factor $p-i\mu$ as a general complex number (that is, p can be any complex number with μ constant). We can plot it in the complex plane like this:



In exponential notation, the angle is measured counterclockwise from the positive x axis, so we have

$$p-i\mu = r_1 e^{i\theta_1 - i\pi/2} \tag{7}$$

The square root of this is

$$\sqrt{p-i\mu} = \sqrt{r_1} e^{i(\theta_1-\pi/2)/2} \tag{8}$$

where $\sqrt{r_1}$ is taken to be the positive square root.

If we rotate θ_1 by 2π, we get the same numerical value for $p-i\mu$, so that

$$p-i\mu = r_1 e^{i\theta_1 - i\pi/2 + 2\pi i} \tag{9}$$

However, the square root picks up a minus sign, since

$$\sqrt{p-i\mu} = \sqrt{r_1} e^{i(\theta_1-\pi/2)/2} e^{\pi i} \tag{10}$$

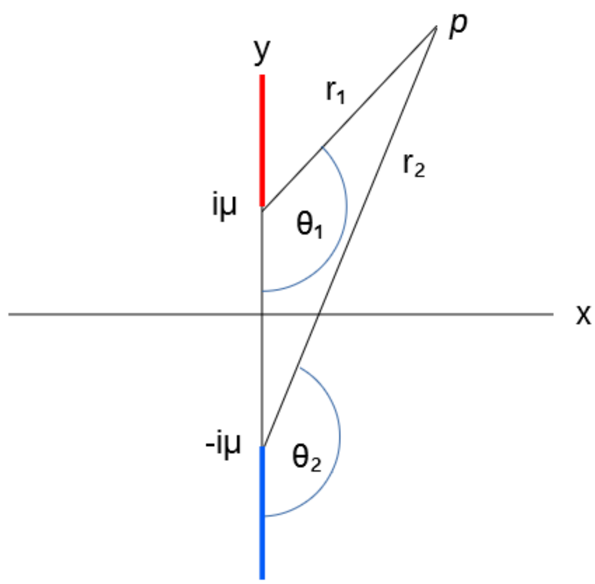
$$= -\sqrt{r_1} e^{i(\theta_1-\pi/2)/2} \tag{11}$$

Thus the origin is a branch point of $\sqrt{p-i\mu}$.

Using a similar argument, we find that the origin is also a branch point for the other factor $\sqrt{p+i\mu}$, whose argument we can write as

$$p+i\mu = r_2 e^{i\theta_2 - i\pi/2} \tag{12}$$

If we shift $p-i\mu$ by a distance μ up the positive y axis and $p+i\mu$ down by a distance μ along the y axis, then the branch points move to $\pm i\mu$ and we get the situation as shown:



We can define a branch cut either along the line segment connecting $-i\mu$ to $+i\mu$, or, as we do here, we can connect these two points by wrapping around at infinity, so we get an upper branch cut (red) extending from $i\mu$ upwards along the y axis to infinity and a lower branch cut (blue) extending from $-i\mu$ downwards to $-\infty$. As long as we don't cross a branch cut, the function 6 is single valued.

We can avoid crossing a branch cut by restricting θ_1 to $-\pi \leq \theta_1 \leq \pi$ and θ_2 to $0 \leq \theta_2 \leq 2\pi$.

Using 7 and 12, we have

$$(p+i\mu)(p-i\mu) = r_1 r_2 e^{i(\theta_1+\theta_2)-i\pi} \tag{13}$$

The square root thus has the value

$$\sqrt{(p+i\mu)(p-i\mu)} = \sqrt{r_1 r_2} e^{i(\theta_1+\theta_2)/2} e^{-i\pi/2} \tag{14}$$

$$= -i\sqrt{r_1 r_2} e^{i(\theta_1+\theta_2)/2} \tag{15}$$

The contour used by Coleman to evaluate the integral 5 lies in the upper half plane and runs down the LHS of the red branch cut and up the RHS of the same branch cut. See here for an example of integrating around a branch cut. Thus we need the values of $\sqrt{(p+i\mu)(p-i\mu)}$ along these two parts of the contour.

Along the LHS of the red branch cut, $\theta_1=-\pi$ and $\theta_2=+\pi$, so we have

$$\left.\sqrt{(p+i\mu)(p-i\mu)}\right|_{LHS} = -i\sqrt{r_1r_2}e^{i(-\pi+\pi)/2} \tag{16}$$

$$= -i\sqrt{r_1r_2} \tag{17}$$

If we write $p=iy$ then

$$\left.\sqrt{(p+i\mu)(p-i\mu)}\right|_{LHS} = \sqrt{-y^2+\mu^2} \tag{18}$$

$$= \pm i\sqrt{y^2-\mu^2} \tag{19}$$

From 17 we see that we must choose the minus sign, so on the LHS of the cut

$$\left.\sqrt{(p+i\mu)(p-i\mu)}\right|_{LHS} = -i\sqrt{y^2-\mu^2} \tag{20}$$

On the RHS of the cut, we have $\theta_1=+\pi$ and $\theta_2=+\pi$, so we have

$$\left.\sqrt{(p+i\mu)(p-i\mu)}\right|_{RHS} = -i\sqrt{r_1r_2}e^{i(\pi+\pi)/2} \tag{21}$$

$$= +i\sqrt{r_1r_2} \tag{22}$$

Therefore

$$\left.\sqrt{(p+i\mu)(p-i\mu)}\right|_{RHS} = i\sqrt{y^2-\mu^2} \tag{23}$$

Coleman uses these results to transform the integral 5 to the form

$$\left\langle \boldsymbol{x}\left|e^{-iHt}\right|\psi\right\rangle = \frac{i}{2\pi^2 r}\int_\mu^\infty dy\; y\; e^{-ry}\sinh\left(\sqrt{y^2-\mu^2}t\right) \tag{24}$$

The key point is that the integrand is entirely positive over the range of integration, so the integral is non-zero for $r>t$. In other words, there is a non-zero probability amplitude for the particle to travel faster than light, although this probability is fairly small.

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